Absence of ground state for the Nelson model on static space-times

Christian Gérard, Fumio Hiroshima, Annalisa Panati, A. Suzuki

ABSENCE OF GROUND STATE
FOR THE NELSON MODEL ON STATIC SPACE-TIMES

C. GÉRARD, F. HIROSHIMA, A. PANATI, AND A. SUZUKI

Abstract. We consider the Nelson model on some static space-times and investigate the problem of absence of a ground state. Nelson models with variable coefficients arise when one replaces in the usual Nelson model the flat Minkowski metric by a static metric, allowing also the boson mass to depend on position. We investigate the absence of a ground state of the Hamiltonian in the presence of the infrared problem, i.e. assuming that the boson mass $m(x)$ tends to 0 at spatial infinity. Using path space techniques, we show that if $m(x) \leq C|x|^{-\mu}$ at infinity for some $C > 0$ and $\mu > 1$ then the Nelson Hamiltonian has no ground state.

1. Introduction

In this paper we continue the study of the so-called *Nelson model with variable coefficients* began in [GHPS1, GHPS2]. The Nelson model with variable coefficients describes a system of quantum particles linearly coupled to a scalar quantum field with an ultraviolet cutoff. Typically the scalar field is the Klein-Gordon field on a static Lorentzian manifold, (see [GHPS2]).

In this respect the Nelson model with variable coefficients is an extension of the standard Nelson model introduced by [N] to the case when the Minkowskian space-time is replaced by a static Lorentzian manifold.

The Hamiltonian of the Nelson model with variable coefficients is defined as a selfadjoint operator on $L^2(\mathbb{R}^3, d\mathrm{x}) \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^3, dx))$, formally given by

$$
\begin{aligned}
H = & -\frac{1}{2} \sum_{1\leq j,k\leq 3} \partial_{\mathrm{x}_j} A^{jk}(\mathrm{x}) \partial_{\mathrm{x}_k} + V(\mathrm{x}) \\
& + \frac{1}{2} \int \big(\pi(x)^2 + \varphi(x)\omega^2(x, D_x)\varphi(x)\big)\, dx \\
& + \frac{q}{\sqrt{2}} \int \omega^{-1/2}(x, D_x)\rho(x - \mathrm{x})\varphi(x) dx,
\end{aligned} \tag{1.1}
$$

where $\varphi(x)$ is the time-zero scalar field, $\pi(x)$ its conjugate momentum, $q \in \mathbb{R}$ a coupling constant, ρ a non-negative cutoff function, and $\omega(x, D_x) = h^{\frac{1}{2}}$ with

$$
h = -c(x)^{-1} \left(\sum_{1\leq j,k\leq 3} \partial_{x_j} a^{jk}(x) \partial_{x_k} \right) c(x)^{-1} + m^2(x). \tag{1.2}
$$

Here $m^2(x)$ describes a variable mass. The assumptions on a^{jk}, A^{jk} and c will be given later in Section 2. We refer to [GHPS2] for the derivation of (1.1) starting from the Lagrangian of a Klein-Gordon field on a static space-time linearly coupled to a non-relativistic particle.

Date: November 2010.

2000 *Mathematics Subject Classification.* 81T10, 81T20, 81Q10, 58C40.

Key words and phrases. Quantum field theory, Nelson model, static space-times, ground state, Feynman-Kac formula.

The standard Nelson model is defined by taking $\omega(x, D_x) = \omega(D_x)$ for $\omega(k) = (k^2 + m^2)^{\frac{1}{2}}$ with a constant $m \geq 0$, and $A^{jk} = \delta_{jk}$. Then $m > 0$ (resp. $m = 0$) corresponds to the massive (resp. massless) case. The model is called *infrared singular* (resp. *regular*) if

$$\int_{\mathbb{R}^3} \frac{|\hat{\rho}(k)|^2}{\omega(k)^3} dk = \infty \text{ (resp. } < \infty),$$

in particular the massive case is always infrared regular. In this paper we will assume that $\rho \geq 0$ and $\int_{\mathbb{R}^3} \rho(x)dx = 1$, which in the standard Nelson model leads to an infrared singular interaction (see Remark 2.4). In the infrared regular case, it is now well known that the standard Nelson Hamiltonian has a unique ground state, see [BFS, DG1, GGM, G, Sp] and [HHS, HS, P, Sa] for more general results. The ground state properties are discussed in [BHLMS] using path space techniques. It is also known that in the infrared singular case the standard Nelson Hamiltonian has no ground state. See [AHH, H, LMS, DG2].

In [GHPS2] the existence of ground states of H is shown when

$$m(x) \geq C\langle x\rangle^{-1}, \ C > 0, \tag{1.3}$$

where $\langle x\rangle = (1 + |x|^2)^{1/2}$. In this paper we will consider the case

$$m(x) \leq C\langle x\rangle^{-\mu}, \ \mu > 1. \tag{1.4}$$

In [GHPS1], the absence of ground state of the Nelson model (1.1) is proven if (1.4) holds for $\mu > 3/2$, for a sufficiently small coupling constant, and $A^{jk}(\mathrm{x}) = \delta_{jk} = a^{jk}(x)$. In the present paper we drastically extend [GHPS1]. In fact we show that if (1.4) holds for some $\mu > 1$ then H has no ground state. Therefore combining the results of [GHPS2] with those of the present paper gives an essentially complete discussion of the problem of existence of a ground state for the Nelson model with variable coefficients.

In [DG2] the absence of ground states for an abstract class of models including the standard Nelson model is shown by making use of the so-called *pull-through formula*. This method does not seem to be applicable in our situation. Instead we use the method developed in [LMS] based on path space arguments. We now briefly explain this approach.

Path space representation of the Nelson model.

One can write the physical Hilbert space $L^2(\mathbb{R}^3) \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^3))$ as $L^2(M, dm)$ for some probability space (M, m) in such a way that the interaction term $\varphi_\rho(\mathrm{x})$ becomes a multiplication operator on M and the semi-group e^{-tH} is positivity improving. Moreover the expectation values $(F|\mathrm{e}^{-tH}G)$ can be written using an appropriate path space measure and a Feynman-Kac formula, and the ground state of the free Hamiltonian H_0, (i.e. H with $q = 0$), is mapped to the constant function $\mathbb{1}$.

The probability space (M, m) and the path space measure are obtained by tensoring the corresponding objects for the particle and field Hamiltonians. For the particle Hamiltonian K we use the fact that K has a strictly positive ground state φ_{p}. We then apply the so called *ground state transform* by unitarily identifying $L^2(\mathbb{R}^3, d\mathrm{x})$ with $L^2(\mathbb{R}^3, \psi_{\mathrm{p}}(\mathrm{x})d\mathrm{x})$, obtaining a new particle Hamiltonian L. One can then construct a diffusion process associated to the semi-group e^{-tL}.

For the field Hamiltonian we use the well-known Gaussian process. The path space representation for the Nelson model is then obtained from a Feynman-Kac-Nelson formula.

Absence of ground state.

After mapping everything to $L^2(Q,\mu)$, an easy argument based on Perron-Frobenius shows that H has no ground state iff

$$\gamma(T) := \frac{(\mathbb{1}|\mathrm{e}^{-TH}\mathbb{1})^2}{(\mathbb{1}|\mathrm{e}^{-2TH}\mathbb{1})}$$

tends to 0 when $T \to +\infty$. Using the Feynman-Kac formula the expectation value $(\mathbb{1}|\mathrm{e}^{-TH}\mathbb{1})$ can be explicitly expressed in terms of the *pair potential* W given by

$$W(\mathrm{x},\mathrm{y},|t|) = (\rho(\cdot-\mathrm{x})|\frac{\mathrm{e}^{-|t|\omega}}{2\omega}\rho(\cdot-\mathrm{y})).$$

The key ingredient to estimate W are Gaussian bounds such as

$$C_1\mathrm{e}^{tC_2\Delta}(x,y) \leq \mathrm{e}^{-t\omega^2}(x,y) \leq C_3\mathrm{e}^{tC_4\Delta}(x,y).$$

By modifying the method used in [LMS, KV] and using the super-exponential decay of ψ_{p}, we can finally show that $\gamma(T) \to 0$ as $T \to \infty$ and we conclude that H has no ground state.

Organization.

This paper is organized as follows. In Section 2 we define the Nelson Hamiltonian with variable coefficients. In Section 3 we consider the semi-groups e^{-tK} and e^{-tL} associated to the two versions of the particle Hamiltonian. We prove the Feynman-Kac formula and various Gaussian bounds on e^{-tK} and e^{-th}. We also construct the diffusion process associated with e^{-tL}. In Section 4 the functional integral representation of e^{-tH} is given. In Section 5 we prove the absence of ground state. Finally Appendix A is devoted to the proof of Proposition 3.12 about the diffusion process associated with L.

2. The Nelson model with variable coefficients

In this section we define the Nelson model with variable coefficients and state our main theorem.

2.1. Notation.

We collect here some notation used in this paper for reader's convenience.

Hilbert space and operators: The domain of a linear operator A on Hilbert space $\mathcal{H}$ will be denoted by $\mathrm{Dom}A$, and its spectrum by $\sigma(A)$. The set of bounded operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $B(\mathcal{H},\mathcal{K})$ and $B(\mathcal{H},\mathcal{H})$ by $B(\mathcal{H})$ for simplicity. The scalar product on $\mathcal{H}$ is denoted by $(u|v)$. Let $\mathcal{X}$ be a real or complex Hilbert space. If a is a selfadjoint operator on $\mathcal{X}$, we will write $a > 0$ if $a \geq 0$ and $\mathrm{Ker}a = \{0\}$. Note that if $a > 0$ and $s \in \mathbb{R}$, $\|h\|_s = \|a^{-s}h\|_{\mathcal{X}}$ is a norm on $\mathrm{Dom}a^{-s}$. We denote then by $a^s\mathcal{X}$ the completion of $\mathrm{Dom}a^{-s}$ for the norm $\|\ \|_s$. The map a^s extends as a unitary operator from $a^t\mathcal{X}$ to $a^{s+t}\mathcal{X}$. One example of this notation are the familiar Sobolev spaces, where $H^s(\mathbb{R}^d)$ is equal to $(-\Delta+1)^{-s/2}L^2(\mathbb{R}^d)$. Finally if $B \in B(L^2(\mathbb{R}^3))$, the distribution kernel of B will be denoted by $B(x,y)$.

Bosonic Fock space: If $\mathfrak{h}$ is a Hilbert space, the *bosonic Fock space* over $\mathfrak{h}$, denoted by $\Gamma_{\mathrm{s}}(\mathfrak{h})$, is

$$\Gamma_{\mathrm{s}}(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n \mathfrak{h}.$$

$\Omega = (1,0,0,\cdots) \in \Gamma_{\mathrm{s}}(\mathfrak{h})$ is called the Fock vacuum. We denote by $a^*(h)$ and $a(h)$ for $h \in \mathfrak{h}$ the *creation* and *annihilation operators*, acting on $\Gamma_{\mathrm{s}}(\mathfrak{h})$. If $\mathcal{K}$ is another Hilbert space and $v \in B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$, then one defines the operators $a^*(v)$, $a(v)$ and

$\phi(v)$ as unbounded operators on $\mathcal{K}\otimes\Gamma_{\rm s}(\mathfrak{h})$ by

$$\begin{aligned}
&a^*(v)\Big|_{\mathcal{K}\otimes(\otimes_{\rm s}^n\mathfrak{h})} := \sqrt{n+1}\Big(\mathbb{1}_{\mathcal{K}}\otimes\mathcal{S}_{n+1}\Big)\Big(v\otimes\mathbb{1}_{\bigotimes_{\rm s}^n\mathfrak{h}}\Big),\\
&a(v) := \big(a^*(v)\big)^*,\\
&\phi(v) := \frac{1}{\sqrt{2}}(a(v)+a^*(v).
\end{aligned}$$

Here $\mathcal{S}_{n+1}$ denotes the symmetrization. If T is a contraction on $\mathcal{H}$, then $\Gamma(T) : \Gamma_{\rm s}(\mathfrak{h})\to\Gamma_{\rm s}(\mathfrak{h})$ is defined as

$$\begin{aligned}
&\Gamma(T)\Big|_{\bigotimes_{\rm s}^n\mathfrak{h}} := \underbrace{T\otimes\cdots\otimes T}_{n},\quad n\geq 1,\\
&\Gamma(T)\Big|_{\bigotimes_{\rm s}^0\mathfrak{h}} := \mathbb{1},\quad n=0.
\end{aligned}$$

If b is a selfadjoint operator on $\mathfrak{h}$, its second quantization $d\Gamma(b) : \Gamma_{\rm s}(\mathfrak{h})\to\Gamma_{\rm s}(\mathfrak{h})$ is defined as

$$\begin{aligned}
&d\Gamma(b)\Big|_{\bigotimes_{\rm s}^n\mathfrak{h}} := \sum_{j=1}^n \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{j-1}\otimes b\otimes\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{n-j},\quad n\geq 1,\\
&d\Gamma(b)\Big|_{\bigotimes_{\rm s}^0\mathfrak{h}} := 0,\quad n=0.
\end{aligned}$$

Let $N=d\Gamma(\mathbb{1})$. The creation operator and the annihilation operators satisfy the estimates

$$\|a^\sharp(v)(N+1)^{-\frac{1}{2}}\|\leq\|v\|, \tag{2.5}$$

where $a^\sharp = a, a^*$ and $\|v\|$ is the norm of v in $B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$.

We denote by $x\in\mathbb{R}^3$ (resp. ${\rm x}\in\mathbb{R}^3$) the boson (resp. particle) position.

2.2. **Particle Hamiltonian.** In this section we define the particle Hamiltonian K on $L^2(\mathbb{R}^3)$. We set

$$K_0 = -\frac{1}{2}\sum_{1\leq j,k\leq 3}\partial_{{\rm x}_j}A^{jk}({\rm x})\partial_{{\rm x}_k},$$

acting on $\mathcal{K}=L^2(\mathbb{R}^3, d{\rm x})$. We assume

$$\begin{aligned}
&(E1)\quad C_0\mathbb{1}\leq[A^{jk}({\rm x})]\leq C_1\mathbb{1},\ C_0>0,\\
&(E2)\quad \nabla_{\rm x}[A^{jk}({\rm x})]\in L^\infty(\mathbb{R}^3).
\end{aligned}$$

In Subsection 3.2 we will consider the drift vector:

$$b({\rm x}) = (b_1({\rm x}), b_2({\rm x}), b_3({\rm x})),\ b_k({\rm x}) = \frac{1}{2}\sum_{j=1}^3\partial_j A^{jk}({\rm x}),$$

and we will need the assumption:

$$(E3)\quad \nabla_{\rm x} b_j({\rm x})\in L^\infty(\mathbb{R}^3).$$

Under assumption (E1), K_0 is defined as the positive selfadjoint operator associated with the closed quadratic form:

$$\mathrm{q}_0(f,f) = \frac{1}{2}\int\sum_{1\leq j,k\leq 3}\overline{\partial_{{\rm x}_j}f({\rm x})}A^{jk}({\rm x})\partial_{{\rm x}_k}f({\rm x})d{\rm x}, \tag{2.6}$$

with form domain $H^1(\mathbb{R}^3)$. Assuming also (E2), then by standard elliptic regularity, we know that

$$K_0 f = -\frac{1}{2}\sum_{1\leq j,k\leq 3}\partial_{{\rm x}_j}(A^{jk}({\rm x})\partial_{{\rm x}_k}f)$$

with $\mathrm{Dom}K_0 = H^2(\mathbb{R}^3)$. We also introduce an external potential V. We assume that

$$(E4)\ V \in L^1_{\mathrm{loc}}(\mathbb{R}^3), V \geq 0.$$

The operator

$$K := K_0 \dot{+} V$$

is defined as the positive selfadjoint operator associated with the closed quadratic form:

$$\mathrm{q}(f,f) = \mathrm{q}_0(f,f) + \int V(\mathrm{x})|f|^2(\mathrm{x})d\mathrm{x},$$

with form domain $H^1(\mathbb{R}^3) \cap \mathrm{Dom}V^{\frac{1}{2}}$. If we assume the following *confining condition:*

$$(E5)\quad b_0\langle \mathrm{x}\rangle^{2\delta} \leq V(\mathrm{x}),\ b_0 > 0,\ \delta > 0.$$

then K has compact resolvent.

2.3. **Boson one-particle energy.** Next we define boson one-particle Hamiltonian. Let

$$\begin{aligned} h_0 := &\quad -c(x)^{-1}\left(\sum_{1\leq j,k\leq d} \partial_j a^{jk}(x)\partial_k\right)c(x)^{-1}, \\ h := &\quad h_0 + m^2(x), \end{aligned} \tag{2.7}$$

where a^{jk}, c, m are real functions and

$$(B1)\quad \begin{aligned} & C_0\mathbb{1} \leq [a^{jk}(x)] \leq C_1\mathbb{1},\ C_0 \leq c(x) \leq C_1,\ C_0 > 0, \\ & \partial_x^\alpha a^{jk}(x) \in O(\langle x\rangle^{-1}),\ |\alpha| \leq 1, \\ & \partial_x^\alpha c(x) \in O(1),\ |\alpha| \leq 2, \\ & \partial_x^\alpha m(x) \in O(1),\ |\alpha| \leq 1. \end{aligned}$$

We assume that the variable mass term $m(x)$ decays at infinity faster than $\langle x\rangle^{-1}$:

$$(B2)\ m(x) \in O(\langle x\rangle^{-\mu}),\ \mu > 1.$$

Clearly h is selfadjoint on $H^2(\mathbb{R}^3)$ and $h \geq 0$. The *one-particle space* and *one-particle energy* are

$$\mathfrak{h} := L^2(\mathbb{R}^3, dx), \quad \omega := h^{\frac{1}{2}}. \tag{2.8}$$

By [GHPS2, Lemma 3.1] we know that

$$\mathrm{Ker}\omega = \{0\},\ \mathrm{inf}\sigma(\omega) = 0.$$

2.4. **Nelson Hamiltonians.** We fix a *charge density* $\rho : \mathbb{R}^3 \to \mathbb{R}$ such that

$$(B3)\ \rho(x) \geq 0,\ \int \rho(x)dx = 1,\ |k|^{-\alpha}\hat{\rho}(k) \in L^2(\mathbb{R}^3, dk),\ \alpha = 1, \frac{1}{2}.$$

where $\hat{\rho}$ denotes the Fourier transform of ρ, and set $\rho_{\mathrm{x}}(x) = \rho(x - \mathrm{x})$. We define the *UV cutoff fields* as

$$\varphi_\rho(\mathrm{x}) := \phi(\omega^{-\frac{1}{2}}\rho_{\mathrm{x}}). \tag{2.9}$$

Note that setting $\varphi(\mathrm{x}) := \phi(\omega^{-\frac{1}{2}}\delta_{\mathrm{x}})$, one has $\varphi_\rho(\mathrm{x}) = \int \varphi(\mathrm{x}-\mathrm{y})\rho(\mathrm{y})d\mathrm{y}$. The *Nelson Hamiltonian* is

$$H := K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + q\varphi_\rho(\mathrm{x}), \tag{2.10}$$

acting on

$$\mathcal{H} = \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathfrak{h}). \tag{2.11}$$

The constant q has the interpretation of the charge of the particle. We assume of course that $q \neq 0$. We also set

$$H_0 := K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega),$$

which is selfadjoint on $\mathrm{Dom}H_0 = \mathrm{Dom}(K \otimes \mathbb{1}) \cap \mathrm{Dom}(\mathbb{1} \otimes \mathrm{d}\Gamma(\omega))$.

Proposition 2.1. *Assume hypotheses (E1), (E4), (B1), (B2), (B3). Then H is selfadjoint and bounded below on* $\mathrm{Dom}H_0$. *Moreover H is essentially selfadjoint on any core of H_0.*

Proof. Since $-\Delta_x \leq C\omega^2$ it follows from the Kato-Heinz theorem that

$$\sup_{\mathrm{x}\in\mathbb{R}^3} \|\omega^{-\alpha}\rho_{\mathrm{x}}\|_{L^2} \leq C\||k|^{-\alpha}\hat{\rho}\|_{L^2},\ \alpha = \frac{1}{2}, 1.$$

It follows e.g., from [GGM, Section 4] that $\varphi_\rho(\mathrm{x})$ is H_0 bounded with the infinitesimal bound, and the proposition follows from the Kato-Rellich theorem. □

Remark 2.2. *In the previous paper* [GHPS1] *we considered the case* $\omega = (-\Delta_x + m^2(x))^{\frac{1}{2}}$ *but with φ_ρ replaced by $\tilde{\varphi}_\rho$ given by*

$$\tilde{\varphi}_\rho(\mathrm{x}) = \phi(\omega^{-\frac{1}{2}}\tilde{\rho}_{\mathrm{x}}), \tag{2.12}$$

where

$$\tilde{\rho}_{\mathrm{x}}(\cdot) = (2\pi)^{-3/2}\int \Psi(k,\cdot)\overline{\Psi(k,\mathrm{x})}\hat{\rho}(k)dk, \tag{2.13}$$

and the generalized eigenfunctions $\Psi(k,x)$ are solutions to the Chapman-Kolmogorov equation:

$$\Psi(k,x) = \mathrm{e}^{\mathrm{i}k\cdot x} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}|k||x-y|}m^2(y)\Psi(k,y)}{|x-y|}dy. \tag{2.14}$$

Note that if ρ is radial e.g., $\rho(x) = \rho(|x|)$ then $\tilde{\varphi}_\rho(x) = \phi(\omega^{-\frac{1}{2}}\hat{\rho}(\omega)\delta_{\mathrm{x}})$. If $m(x) \equiv 0$ then $\tilde{\varphi}_\rho(\mathrm{x}) = \varphi_\rho(\mathrm{x})$. In the general case, $\omega = h^{\frac{1}{2}}$, the natural definition (2.9) is much more convenient than (2.12). In particular we do not need to consider generalized eigenfunctions for h defined in (2.7).

2.5. **Absence of ground state for Nelson Hamiltonians.** The main theorem in this paper is as follows:

Theorem 2.3. *Assume hypotheses (E1), (E2), (E3), (E5), (B1), (B2) and (B3). Then H has no ground state.*

Remark 2.4. *Since $\hat{\rho}(0) = 1$, we see that*

$$\int_{\mathbb{R}^3} \frac{|\hat{\rho}(k)|^2}{|k|^3}dk = \infty. \tag{2.15}$$

As is well known if $\omega = (-\Delta_x)^{\frac{1}{2}}$, (2.15) is called the infrared singular condition*. In this case Theorem 2.3 is well known, see e.g.,* [DG2].

Remark 2.5. *In* [GHPS2] *we show that if instead of (B2) we assume that $m(x) \geq C\langle x\rangle^{-1}$ then H as a (unique) ground state. Therefore Theorem 2.3 is sharp with respect to the decay rate of the mass at infinity.*

3. Feynman-Kac formula for the particle Hamiltonian

In this section we prove some Gaussian bounds on the heat kernels e^{-tK_0}, e^{-th_0} and e^{-th}. The bounds for e^{-tK_0} and e^{-th_0} are well known in various contexts. In our situation they are due to [PE, Theorems 3.4 and 3.6]. Note that by identifying x and x and setting $c(x) \equiv 1$, K_0 is a particular case of h_0.

3.1. **Gaussian upper and lower bounds on heat kernels.** The bounds for e^{-th} were proved previously by [Se] for operators in divergence form and by [Zh] for Laplace-Beltrami operators.

Proposition 3.1. [PE, Theorems 3.4 and 3.6] *Assume (B1), or (E1), (E2). Then there exist constants $C_i,\ c_i > 0$ such that*

$$C_1 \mathrm{e}^{c_1 t\Delta}(x,y) \leq \mathrm{e}^{-th_0}(x,y) \leq C_2 \mathrm{e}^{c_2 t\Delta}(x,y),\ \forall\ t>0,\ x,y\in\mathbb{R}^3, \tag{3.16}$$

$$C_1 \mathrm{e}^{c_1 t\Delta}(\mathrm{x},\mathrm{y}) \leq \mathrm{e}^{-tK_0}(\mathrm{x},\mathrm{y}) \leq C_2 \mathrm{e}^{c_2 t\Delta}(\mathrm{x},\mathrm{y}),\ \forall\ t>0,\ \mathrm{x},\mathrm{y}\in\mathbb{R}^3. \tag{3.17}$$

Proposition 3.2. *Assume (B1) and (B2). Then there exist constants $C_i, c_i > 0$ such that*

$$C_1 \mathrm{e}^{c_1 t\Delta}(x,y) \leq \mathrm{e}^{-th}(x,y) \leq C_2 \mathrm{e}^{c_2 t\Delta}(x,y),\ \forall\ t>0,\ x,y\in\mathbb{R}^d.$$

Remark 3.3. *Conjugating by the unitary*

$$U : L^2(\mathbb{R}^d, dx) \ni u \mapsto c(x)^{-1}u \in L^2(\mathbb{R}^d, c^2(x)dx),$$

we obtain

$$\begin{aligned} \tilde{h}_0 :=&\ \ U h_0 U^{-1} = -c(x)^{-2} \textstyle\sum_{1\leq j,k\leq d} \partial_j a^{jk}(x)\partial_k, \\ \tilde{h} :=&\ \ U h U^{-1} = \tilde{h}_0 + m^2(x), \end{aligned}$$

which are selfadjoint with domain $H^2(\mathbb{R}^d)$. Let $\mathrm{e}^{-t\tilde{h}}(x,y)$ for $t>0$ the integral kernel of $\mathrm{e}^{-t\tilde{h}}$ i.e. such that

$$\mathrm{e}^{-t\tilde{h}}u(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-t\tilde{h}}(x,y)u(y)c^2(y)dy,\ t>0.$$

Then since $\mathrm{e}^{-th}(x,y) = c(x)\mathrm{e}^{-t\tilde{h}}(x,y)c(y)$, the bounds in Proposition 3.1 also hold for $\tilde{h}_0$ and it suffices to prove Proposition 3.2 for $\mathrm{e}^{-t\tilde{h}}$.

By the above remark, we will consider the operators $\tilde{h}_0$ and $\tilde{h}$. We note that they are associated with the closed quadratic forms:

$$\begin{aligned} \tilde{h}_0(f,f) &= \int_{\mathbb{R}^d} \sum_{1\leq j,k\leq 3} \overline{\partial_j f(x)} a^{jk}(x)\partial_k f(x)\ dx, \\ \tilde{h}(f,f) &= \tilde{h}_0(f,f) + \int_{\mathbb{R}^d} |f|^2(x)m^2(x)c^2(x)\ dx, \end{aligned} \tag{3.18}$$

with domain $H^1(\mathbb{R}^d)$. We will use the following well known convexity result. For completeness we sketch its proof below.

Lemma 3.4. *Assume that $w \in L^\infty(\mathbb{R}^3)$ is a real potential. Then*

$$\mathbb{R} \ni \lambda \mapsto \mathrm{e}^{-t(\tilde{h}_0+\lambda w)}(x,y)$$

is logarithmically convex for all $t>0$ and a.e. $x,y\in\mathbb{R}^3$.

Proof. Note that if F_1 and F_2 are log-convex, then F_1F_2 and CF_1 are log-convex. Moreover (see [S, Theorem 13.1]) if for all $y\in\mathbb{R}^d$ the function $\mathbb{R}\ni\lambda\mapsto F(\lambda,y)$ is log-convex, so is $\lambda\mapsto\int_{\mathbb{R}^d} F(\lambda,y)dy$. To prove the claim we use the Trotter product formula. We can set $t=1$.

$$\mathrm{e}^{-(\tilde{h}_0+\lambda w)}(x,y) = \lim_{n\to\infty} (\mathrm{e}^{-\tilde{h}_0/n}\mathrm{e}^{-\lambda w/n})^n(x,y),\ \text{a.e. } x,y.$$

Let $A_\lambda(x,y)$, $B_\lambda(x,y)$ the kernels of two operators A_λ, B_λ assumed to be log-convex in λ for a.e. x,y. Then by the above remarks the kernel of $A_\lambda B_\lambda$

$$(A_\lambda B_\lambda)(x,y) = \int_{\mathbb{R}^d} A_\lambda(x,y')B_\lambda(y',y)dy'$$

is also log-convex in λ. The kernel of $\mathrm{e}^{-\tilde{h}_0/n}\mathrm{e}^{-\lambda w/n}$ equals to $\mathrm{e}^{-\tilde{h}_0/n}(x,y)\mathrm{e}^{-\lambda w(y)/n}$ is log-convex in λ. Applying the above remark and the Trotter formula we obtain our claim. □

Proof of Proposition 3.2. We use the unitary transformation as in Remark 3.3. We know from Proposition 3.1 that

$$C_1\mathrm{e}^{c_1t\Delta}(x,y) \leq \mathrm{e}^{-t\tilde{h}_0}(x,y) \leq C_2\mathrm{e}^{c_2t\Delta}(x,y),\ \forall\, t>0,\ x,y\in\mathbb{R}^d. \tag{3.19}$$

Since $m^2(x)\geq 0$, the upper bound in the proposition follows from the Trotter-Kato formula. Let us now prove the lower bound, following the arguments in [Se, Theorem 6.1]. Since $m^2(x)c^2(x)\in O(\langle x\rangle^{-2-\epsilon})$ it follows from Hardy's inequality and (3.18) that

$$\tilde{h}_0 \geq c_0 m^2,\ \text{for some } c_0>0.$$

Set now $w(x)=-c_0m^2(x)/4$. Since $\tilde{h}_0+2w\geq \frac{1}{2}\tilde{h}_0$, we deduce from [D, Theorem 2.4.2] that

$$\|\mathrm{e}^{-t(\tilde{h}_0+2w)}\|_{L^2\to L^\infty}\leq Ct^{-d/4}.$$

By duality this implies that

$$\|\mathrm{e}^{-t(\tilde{h}_0+2w)}\|_{L^1\to L^2}\leq Ct^{-d/4},$$

and hence

$$\|\mathrm{e}^{-t(\tilde{h}_0+2w)}\|_{L^1\to L^\infty}\leq \|\mathrm{e}^{-t(\tilde{h}_0+2w)/2}\|_{L^2\to L^\infty}\|\mathrm{e}^{-t(\tilde{h}_0+2w)/2}\|_{L^1\to L^2}\leq Ct^{-d/2}.$$

By [D, Lemma 2.1.2] we obtain

$$\mathrm{e}^{-t(\tilde{h}_0+2w)}(x,y)\leq Ct^{-d/2}.$$

Applying then Lemma 3.4 this yields

$$\mathrm{e}^{-t(\tilde{h}_0+w)}(x,y)\leq t^{-d/4}\mathrm{e}^{-th_0}(x,y)^{\frac{1}{2}},\ \text{a.e. } x,y\in\mathbb{R}^d. \tag{3.20}$$

Applying once more the log-convexity, we get that

$$\mathrm{e}^{-t\tilde{h}_0}(x,y)\leq \mathrm{e}^{-t(\tilde{h}_0+v)}(x,y)^s\mathrm{e}^{-t(\tilde{h}_0+w)}(x,y)^{1-s},\ \text{for } s=c_0/(4+c_0),$$

and hence using (3.20):

$$\mathrm{e}^{-t\tilde{h}_0}(x,y)^{(1+s)/2}t^{(1-s)d/4}\leq \mathrm{e}^{-t(\tilde{h}_0+m^2c^2)}(x,y)^s,$$

which implies the lemma using the lower bound in (3.19). □

3.2. **Stochastic differential equation.** Recall that we introduced the drift vector $b(\mathrm{x})$ in Subsection 2.2. We also define the diffusion matrix:

$$\sigma(\mathrm{x}):=[A^{jk}]^{\frac{1}{2}}(\mathrm{x}),$$

and note that it follows from (E1), (E2), (E3) that $b(\mathrm{x})$, $\sigma(\mathrm{x})$ are uniformly Lipschitz on $\mathbb{R}^3$. We consider the stochastic differential equation:

$$\left\{\begin{array}{lcl} dX_t^{\mathrm{x}} &=& b(X_t^{\mathrm{x}})dt+\sigma(X_t^{\mathrm{x}})dB_t,\ t\geq 0,\\ X_0^{\mathrm{x}} &=& \mathrm{x}, \end{array}\right. \tag{3.21}$$

on the probability space $(\mathscr{X}_+, B(\mathscr{X}_+), \mathcal{W})$, where $\mathscr{X}_+=C([0,\infty);\mathbb{R}^3)$, $B(\mathscr{X}_+)$ is the σ-field generated by cylinder sets and $\mathcal{W}$ the Wiener measure. $(B_t)_{t\geq 0}$ denotes the 3-dimensional Brownian motion on $(\mathscr{X}_+, B(\mathscr{X}_+), \mathcal{W})$ starting at 0. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration of the Brownian motion: $\mathcal{F}_t=\sigma(B_s, 0\leq s\leq t)$.

Proposition 3.5. *Assume (E1), (E2), (E3). Then (3.21) has the unique solution* $X^{\mathrm{x}} = (X_t^{\mathrm{x}})_{t\geq 0}$ *which is a diffusion process with respect to the filtration* $(\mathcal{F}_t)_{t\geq 0}$*:*

$$\mathbb{E}_{\mathcal{W}}\left[f(X_{s+t}^{\mathrm{x}})|\mathcal{F}_s\right] = \mathbb{E}_{\mathcal{W}}\left[f(X_t^{X_s^{\mathrm{x}}})\right] \tag{3.22}$$

for any bounded Borel measurable function f*, where* $\mathbb{E}_{\mathcal{W}}\left[f(X_t^{X_s^{\mathrm{x}}})\right]$ *is* $\mathbb{E}_{\mathcal{W}}\left[f(X_t^{\mathrm{y}})\right]$ *evaluated at* $\mathrm{y} = X_s^{\mathrm{x}}$*.*

Proof. Since b, σ are bounded and uniformly Lipschitz, the proposition follows from [O, Theorem 5.2.1]. $\Box$

The following proposition is well-known. For lack of a precise reference, we will sketch its proof.

Proposition 3.6. *Assume (E1), (E2), (E3). Then*

$$\mathrm{e}^{-tK_0}f(\mathrm{x}) = \mathbb{E}_{\mathcal{W}}\left[f(X_t^{\mathrm{x}})\right], \quad t\geq 0, \ \textit{a.e. } \mathrm{x}\in\mathbb{R}^3. \tag{3.23}$$

for $f\in L^2(\mathbb{R}^3)$*.*

Proof. We first prove (3.23) for $f\in C_0^\infty(\mathbb{R}^3)$, under the additional assumption that

$$\partial_{\mathrm{x}}^\alpha A^{jk}(\mathrm{x})\in L^\infty(\mathbb{R}^3), \ \forall\, \alpha\in\mathbb{N}^3. \tag{3.24}$$

Let $u(t,\mathrm{x}) := \mathrm{e}^{(t-T)K_0}f(\mathrm{x})$, $0\leq t\leq T$. By elliptic regularity we know that $\mathrm{Dom}K_0^n = H^{2n}(\mathbb{R}^3)$ and using that $u\in C^k([0,T],\mathrm{Dom}K_0^n)$ we see using Sobolev's inequalities that $u(t,\mathrm{x})$ is a bounded $C^{1,2}([0,T]\times\mathbb{R}^3)$ solution of:

$$\partial_t u(t,\mathrm{x}) = K_0 u(t,\mathrm{x}), \ u(T,\mathrm{x}) = f(\mathrm{x}).$$

By [KS, Thm. 7.6] it follows that

$$u(0,\mathrm{x}) = \mathrm{e}^{-TK_0}f(\mathrm{x}) = \mathbb{E}_{\mathcal{W}}\left[f(X_T^{\mathrm{x}})\right],$$

which proves (3.23) in this case.

We assume now only (E1), (E2), (E3). We can find a sequence $[A^{jk}]_n(\mathrm{x})$ satisfying (3.24), such that $[A^{jk}]_n(\mathrm{x})$, $b_n(\mathrm{x})$ are uniformly Lipschitz and

$$[A^{jk}]_n\to[A^{jk}], \ b_n\to b, \ \text{uniformly in } \mathbb{R}^3.$$

This also implies that $\sigma_n^{jk}\to\sigma^{jk}$ uniformly in $\mathbb{R}^3$.

Let us denote by $X_{t,n}^{\mathrm{x}}$ the solution of (3.21) with σ_n, b_n and by $K_{0,n}$ the associated differential operator. By a well known stability result for solutions of stochastic differential equations, see e.g. [E, Chapter 5] we obtain that

$$\mathbb{E}_{\mathcal{W}}\left[|X_{t,n}^{\mathrm{x}}-X_t^{\mathrm{x}}|^2\right]\to 0, \ \forall \mathrm{x}\in\mathbb{R}^3.$$

Let us fix $f\in C_0^\infty(\mathbb{R}^3)$. Taking a sub sequence, we obtain that $f(X_{t,n}^{\mathrm{x}})\to f(X_t^{\mathrm{x}})$ a.e. $\mathcal{W}$ and hence that $T_{t,n}f(\mathrm{x})\to T_tf(\mathrm{x})$. On the other hand we see that $K_{0,n}\to K_0$ in norm resolvent sense, hence $\mathrm{e}^{-tK_{0,n}}f\to\mathrm{e}^{-tK_0}f$ in $L^2(\mathbb{R}^3)$. Taking again a sub sequence, we obtain that $\mathrm{e}^{-tK_{0,n}}f(\mathrm{x})\to\mathrm{e}^{-tK_0}f(\mathrm{x})$ a.e. x. Therefore the identity (3.23) holds for $f\in C_0^\infty(\mathbb{R}^3)$, under assumptions (E1), (E2), (E3).

We first extend (3.23) to $f\in L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ by density. For $t\geq 0$, $f\in C_0^2(\mathbb{R}^3)$ we set

$$m_t(f) = \int\mathbb{E}_{\mathcal{W}}[f(X_t^{\mathrm{x}})]d\mathrm{x}.$$

Clearly $f\geq 0$ implies $m_t(f)\geq 0$. Since K_0 is a uniformly elliptic operator, e^{-tK_0} is a contraction on $L^1(\mathbb{R}^3)$ [D, Theorem 1.3.9]. Using again (3.23) we get

$$|m_t(f)|\leq\int_{\mathbb{R}^3}|f(\mathrm{x})|d\mathrm{x}, \ f\in C_0^2(\mathbb{R}^3), \tag{3.25}$$

and (3.25) can be extended to $f \in L^1$. It also follows from the Riesz-Markov theorem that there exists a Borel measure ϱ_t on $\mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3} f(\mathrm{x}) d\varrho_t(\mathrm{x}) = m_t(f) \tag{3.26}$$

for all $f \in C_0^2(\mathbb{R}^3)$. Together with (3.25) it follows that

$$\left| \int_{\mathbb{R}^3} f(\mathrm{x}) d\varrho_t(\mathrm{x}) \right| \leq \int_{\mathbb{R}^3} |f(\mathrm{x})| d\mathrm{x}. \tag{3.27}$$

Let now $f \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. We can find a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in C_0^2(\mathbb{R}^3)$ such that $f_n \to f$ in L^2, $f_n \to f$ a.e. in $\mathbb{R}^3$ and $\sup_n \|f_n\|_\infty < \infty$. Let us fix $t > 0$. Since $f_n \to f$ in L^2, we get that $e^{-tK_0} f_n \to \mathrm{e}^{-tK_0} f$ in $\mathrm{Dom} K_0 = H^2(\mathbb{R}^3)$, hence uniformly on $\mathbb{R}^3$. Let

$$\begin{aligned} \mathscr{N} = &\quad \{x \in \mathbb{R}^3 | f_n(x) \not\to f(\mathrm{x})\}, \\ \tilde{\mathscr{N}} = &\quad \{(\mathrm{x}, \omega) \in \mathbb{R}^3 \times \mathscr{X}_+ | f_n(X_t^{\mathrm{x}}(\omega)) \not\to f(X_t^{\mathrm{x}}(\omega))\}. \end{aligned}$$

By (3.27) we have

$$\begin{aligned} &\int \mathbb{1}_{\tilde{\mathscr{N}}} d\mathrm{x} \otimes d\mathcal{W} = \int \mathbb{1}_{\mathscr{N}}(X_t^{\mathrm{x}}(\omega)) d\mathrm{x} \otimes d\mathcal{W} \\ &= \int \mathbb{1}_{\mathscr{N}}(\mathrm{x}) d\varrho_t(\mathrm{x}) \leq \int \mathbb{1}_{\mathscr{N}}(\mathrm{x}) d\mathrm{x} = 0, \end{aligned}$$

since $f_n(\mathrm{x}) \to f(\mathrm{x})$ a.e. Hence

$$f_n(X_t^{\mathrm{x}}(\omega)) \to f(X_t^{\mathrm{x}}(\omega)), \text{ a.e. } (\mathrm{x}, \omega) \tag{3.28}$$

with respect to $d\mathrm{x} \otimes d\mathcal{W}$. Therefore using that $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded, we have

$$\mathbb{E}_{\mathcal{W}}[f_n(X_t^{\mathrm{x}})] \to \mathbb{E}_{\mathcal{W}}[f(X_t^{\mathrm{x}})] \text{ a.e. x},$$

which proves (3.23) for $f \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.

Finally let us extend (3.23) to $f \in L^2(\mathbb{R}^3)$. We may assume that $f \geq 0$ without loss of generality. We set $f_n := \min\{f, n\}$, $n \in \mathbb{N}$, so that $f_n \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $f_n(\mathrm{x}) \nearrow f(\mathrm{x})$. Since e^{-tK_0} is positivity preserving, we see that

$$(\mathrm{e}^{-tK_0} f_n)(\mathrm{x}) \nearrow (\mathrm{e}^{-tK_0} f)(\mathrm{x}) < \infty, \text{ a.e. x}.$$

By the same argument as above we get

$$f_n(X_t^{\mathrm{x}}(\omega)) \nearrow f(X_t^{\mathrm{x}}(\omega)), \text{ a.e. } (\mathrm{x}, \omega),$$

and applying (3.23) to f_n we see that $\sup_n \mathbb{E}_{\mathcal{W}}[f_n(X_t^{\mathrm{x}})] < \infty$ a.e. x. The monotone convergence theorem yields that $\mathbb{E}_{\mathcal{W}}[f_n(X_t^{\mathrm{x}})] \nearrow \mathbb{E}_{\mathcal{W}}[f(X_t^{\mathrm{x}})] < \infty$ a.e. x, which completes the proof of the proposition. □

3.3. **Feynman-Kac formula.**

Proposition 3.7. (Feynman-Kac type formula) *Let $f \in L^2(\mathbb{R}^3)$. Assume (E1), (E2), (E3), (E4). Then*

$$\left(\mathrm{e}^{-tK} f\right)(\mathrm{x}) = \mathbb{E}_{\mathcal{W}}\left[f(X_t^{\mathrm{x}}) \mathrm{e}^{-\int_0^t V(X_s^{\mathrm{x}}) ds}\right]. \tag{3.29}$$

Proof. We assume for simplicity that V is continuous. The extension to $V \in L^1_{\mathrm{loc}}$, $V \geq 0$ can be done by the same argument as in e.g. [S, Thm. 6.2]. By the Trotter-Kato product formula [KM] we have

$$\mathrm{e}^{-tK} f = \lim_{n\to\infty} (\mathrm{e}^{-(t/n)K_0} \mathrm{e}^{-(t/n)V})^n f. \tag{3.30}$$

Let $0 \le s_i \in \mathbb{R}$, $f_i \in L^\infty(\mathbb{R}^3)$ for $1 \le i \le n$. By Proposition 3.6 we have:

$$\begin{aligned}
&\left(\mathrm{e}^{-s_1 K_0} f_1 \cdots \mathrm{e}^{-s_n K_0} f_n\right)(\mathrm{x}) \\
&= \mathbb{E}_{\mathcal{W}}\left[f_1(X^{\mathrm{x}}_{s_1})\left(\mathrm{e}^{-s_2 K_0} f_1 \cdots \mathrm{e}^{-s_n K_0} f_n\right)(X^{\mathrm{x}}_{s_1})\right] \\
&= \mathbb{E}_{\mathcal{W}}\left[f_1(X^{\mathrm{x}}_{s_1})\mathbb{E}_{\mathcal{W}}\left[f_2(X^{X^{\mathrm{x}}_{s_1}}_{s_2})\left(\mathrm{e}^{-s_3 K_0} f_3 \cdots \mathrm{e}^{-s_n K_0} f_n\right)(X^{X^{\mathrm{x}}_{s_1}}_{s_2})\right]\right].
\end{aligned}$$

By the Markov property (3.22) we also have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{W}}\left[f_1(X^{\mathrm{x}}_{s_1})\mathbb{E}_{\mathcal{W}}\left[f_2(X^{X^{\mathrm{x}}_{s_1}}_{s_2})\left(\mathrm{e}^{-s_3 K_0} f_3 \cdots \mathrm{e}^{-s_n K_0} f_n\right)(X^{X^{\mathrm{x}}_{s_1}}_{s_2})\right]\right] \\
&= \mathbb{E}_{\mathcal{W}}\left[f_1(X^{\mathrm{x}}_{s_1})\mathbb{E}_{\mathcal{W}}\left[f_2(X^{\mathrm{x}}_{s_1+s_2})\left(\mathrm{e}^{-s_3 K_0} f_3 \cdots \mathrm{e}^{-s_n K_0} f_n\right)(X^{\mathrm{x}}_{s_1+s_2})\,|\mathcal{F}_{s_1}\right]\right] \\
&= \mathbb{E}_{\mathcal{W}}\left[f_1(X^{\mathrm{x}}_{s_1}) f_2(X^{\mathrm{x}}_{s_1+s_2})\left(\mathrm{e}^{-s_3 K_0} f_3 \cdots \mathrm{e}^{-s_n K_0} f_n\right)(X^{\mathrm{x}}_{s_1+s_2})\right].
\end{aligned}$$

Inductively we obtain that
(3.31)

$$\left(\mathrm{e}^{-s_1 K_0} f_1 \cdots \mathrm{e}^{-s_n K_0} f_n\right)(\mathrm{x}) = \mathbb{E}_{\mathcal{W}}\left[\prod_{j=1}^{n} f_j(X^{\mathrm{x}}_{t_j})\right], \text{ for } t_1 = s_1,\ t_j = t_{j-1} + s_j.$$

Combining the Trotter product formula (3.30) and (3.31) with $s_j = t/n$, $f_j = \mathrm{e}^{-(t/n)V}$ we have

$$\mathrm{e}^{-tK} f(\mathrm{x}) = \lim_{n\to\infty} \mathbb{E}_{\mathcal{W}}\left[\mathrm{e}^{-(t/n)\sum_{j=1}^{n} V(X^{\mathrm{x}}_{tj/n})} f(X^{\mathrm{x}}_t)\right]. \tag{3.32}$$

Since $t \to X^{\mathrm{x}}_t$ is continuous a.e. $\mathcal{W}$ and V is continuous it follows that

$$(t/n)\sum_{j=1}^{n} V(X^{\mathrm{x}}_{tj/n}) \to \int_0^t V(X^{\mathrm{x}}_s)ds, \text{ a.e. } \mathcal{W} \text{ when } n \to \infty.$$

Using that $V(\mathrm{x}) \ge 0$ and the Lebesgue dominated convergence theorem, we obtain that

$$\mathbb{E}_{\mathcal{W}}\left[\mathrm{e}^{-(t/n)\sum_{j=1}^{n} V(X^{\mathrm{x}}_{tj/n})} f(X^{\mathrm{x}}_t)\right] \to \mathbb{E}_{\mathcal{W}}\left[\mathrm{e}^{-\int_0^t V(X^{\mathrm{x}}_s ds)} f(X^{\mathrm{x}}_t)\right] \text{ in } L^2(\mathbb{R}^3).$$

This completes the proof of the proposition. □

3.4. **Bounds on heat kernels.** We first recall some easy consequences of the Feynman-Kac formula.

Proposition 3.8. *Assume (E1), (E2), (E3), (E4). Then there exist constants* $C, c > 0$ *such that*

$$\mathrm{e}^{-tK}(\mathrm{x},\mathrm{y}) \le c\mathrm{e}^{Ct\Delta}(\mathrm{x},\mathrm{y}),\ t \ge 0, \quad a.e.\ \mathrm{x},\mathrm{y} \in \mathbb{R}^3. \tag{3.33}$$

Here

$$\mathrm{e}^{T\Delta}(\mathrm{x},\mathrm{y}) = (4\pi T)^{-3/2}\mathrm{e}^{-|\mathrm{x}-\mathrm{y}|^2/(4T)}$$

is the three dimensional heat kernel.

Proof. By the Feynman-Kac formula we know that

$$\mathrm{e}^{-tK}(\mathrm{x},\mathrm{y}) \le \mathrm{e}^{-tK_0}(\mathrm{x},\mathrm{y}), \quad t \ge 0, \quad \text{a.e. } \mathrm{x},\mathrm{y} \in \mathbb{R}^3.$$

Then we apply Proposition 3.1. □

Using the upper bound in Proposition 3.8, we get the following corollary.

Corollary 3.9. (Ultracontractivity) *Assume (E1), (E2), (E3), (E4). Then* e^{-tK} *maps* $L^2(\mathbb{R}^3)$ *to* $L^\infty(\mathbb{R}^3)$ *for* $t > 0$.

Corollary 3.10. (Positivity improving) *Assume (E1), (E2), (E3), (E4). Then* e^{-tK} *is positivity improving for* $t > 0$. *In particular if (E5) holds* K *has a unique strictly positive ground state.*

Proof.
We first claim that

$$\int_0^t V(X_s^{\mathrm{x}})ds < \infty, \text{ a.e.}(x,\omega). \tag{3.34}$$

Assume first that $V \in L^1(\mathbb{R}^3)$. Then since e^{-sK} are contractions on L^1 we get:

$$\int_{\mathbb{R}^3} d\mathrm{x}\mathbb{E}_{\mathcal{W}}\left[\int_0^t V(X_s^{\mathrm{x}})ds\right] = \int_0^t (1|\mathrm{e}^{-sK}V)ds \leq t\|V\|_1,$$

hence (3.34) holds for $V \in L^1(\mathbb{R}^3)$. If $V \in L^1_{\mathrm{loc}}(\mathbb{R}^3$, then $V_n := \mathbb{1}_{\{|x|\leq n\}}V \in L^1(\mathbb{R}^3)$ and there exist sets $\mathcal{N}_n \in \mathbb{R}^3 \times \mathscr{X}_+$ of measure zero such that

$$\int_0^t V_n(X_s^{\mathrm{x}})ds < \infty, \ (x,\omega) \in \mathcal{N}_n.$$

Set $\mathcal{N} := \bigcup_{n\geq 1}\mathcal{N}_n$. Since $s \mapsto X_s^{\mathrm{x}}(\omega)$ is continuous, for each (x,ω) there exists $N = N(\mathrm{x},\omega) \in \mathbb{N}$ such that $N \geq \sup_{0\leq s\leq t}|X_s^{\mathrm{x}}(\omega)|$ and hence $V(X_s^{\mathrm{x}}(\omega)) = V_N(X_s^{\mathrm{x}}(\omega))$ for all $0 \leq s \leq t$. Therefore

$$\int_0^t V(X_s^{\mathrm{x}})ds < \infty, (x,\omega) \notin \mathcal{N},$$

which proves (3.34). To prove that e^{-tK} is positivity improving it suffices to prove that for $f, g \geq 0$ with $f, g \not\equiv 0$ one has $(f|e^{-tK}g) > 0$. Assume that

$$(f|\mathrm{e}^{-tK}g) = \int_{\mathbb{R}^3} d\mathrm{x}\mathbb{E}_{\mathcal{W}}\left[f(\mathrm{x})g(X_t^{\mathrm{x}})\mathrm{e}^{-\int_0^t V(X_s^{\mathrm{x}})ds}\right] = 0. \tag{3.35}$$

It follows from (3.34) that $\mathrm{e}^{-\int_0^t V(X_s^{\mathrm{x}})ds} > 0$ a.e. (x,ω). Hence (3.35) implies that

$$\int_{\mathbb{R}^3} d\mathrm{x}\mathbb{E}_{\mathcal{W}}\left[f(\mathrm{x})g(X_t^{\mathrm{x}})\right] = (f|e^{-tK_0}g) = 0.$$

But this contradicts the lower bound in Prop. 3.1. □

Lemma 3.11. (Exponential decay) *Assume (E1), (E2), (E3), (E5). Let ψ_{p} be the unique strictly positive ground state of K. Then there $\delta > 0$ such that*

$$\mathrm{e}^{\delta|\mathrm{x}|^{\delta+1}}\psi_{\mathrm{p}} \in H^1(\mathbb{R}^3).$$

Proof. If $F \in C^\infty(\mathbb{R}^3)$ is real, bounded with all derivatives, then for $u \in \mathrm{Dom}K$ we have the well-known Agmon identity:

$$\begin{aligned}&\int \frac{1}{2}\langle\nabla(\mathrm{e}^F\overline{u}), A\nabla(\mathrm{e}^F u)\rangle d\mathrm{x} + \int \mathrm{e}^{2F}(V - \frac{1}{2}\langle\nabla F, A\nabla F\rangle)|u|^2 d\mathrm{x}\\ &= \int \mathrm{e}^{2F}\overline{u}Kud\mathrm{x} + 2\mathrm{i}\mathrm{Im}\int \mathrm{e}^{2F}\langle\nabla\overline{u}, A\nabla F\rangle ud\mathrm{x}.\end{aligned}$$

Applying this identity to the real function ψ_{p}, we obtain by the usual argument that there exists $\delta > 0$ such that $\mathrm{e}^{\delta\langle\mathrm{x}\rangle^{\delta+1}}\psi_{\mathrm{p}} \in L^2(\mathbb{R}^3)$ and $\nabla(\mathrm{e}^{\delta\langle\mathrm{x}\rangle^{\delta+1}}\psi_{\mathrm{p}}) \in L^2(\mathbb{R}^3)$.
□

3.5. **Ground state transformation and diffusion process.** Assume (E1), (E2), (E3) and (E5). Then K has compact resolvent and by Corollary 3.10 it has a unique normalized strictly positive ground state ψ_{p}. We set

$$d\mu_{\mathrm{p}}(\mathrm{x}) = \psi_{\mathrm{p}}^2(\mathrm{x})d\mathrm{x}, \ \mathcal{H}_{\mathrm{p}} = L^2(\mathbb{R}^3, d\mu_{\mathrm{p}}), \tag{3.36}$$

and introduce the *ground state transformation:*

$$\mathscr{U}_p : \mathcal{H}_{\mathrm{p}} \to L^2(\mathbb{R}^3), \quad f \mapsto \psi_{\mathrm{p}}f.$$

Let L be the corresponding transform of $K-\inf\sigma(K)$ defined by

$$L = \mathscr{U}_p^{-1}(K-\inf\sigma(K))\mathscr{U}_p \tag{3.37}$$

with $\mathrm{Dom}(L)=\mathscr{U}_p^{-1}\mathrm{Dom}(K)$. We note that

$$(f|Lg)_{\mathcal{H}_\mathrm{p}} = (\psi_\mathrm{p} f|K\psi_\mathrm{p} g)_{L^2} - \inf\sigma(K)(\psi_\mathrm{p} f,|\psi_\mathrm{p} g)_{L^2}.$$

Our goal in this subsection is to construct a three dimensional *diffusion process* (i.e., a continuous Markov process) $X=(X_t)_{t\in\mathbb{R}}$ associated with L. The operator L is formally of the form

$$L = -\frac{1}{2}\sum_{1\le j,k\le 3} A^{jk}\partial_{\mathrm{x}_j}\partial_{\mathrm{x}_k} + \sum_{1\le j,k\le 3}\left(\frac{1}{2}(\partial_{\mathrm{x}_j}A^{jk}) + A^{jk}\frac{\partial_{\mathrm{x}_j}\psi_\mathrm{p}}{\psi_\mathrm{p}}\right)\partial_{\mathrm{x}_k}. \tag{3.38}$$

A standard way to construct the diffusion process X_t is to solve the following stochastic differential equation:

$$dX_t^j = \sum_{k=1}^3 \sigma^{jk}(X_t)dB_t^k + \sum_{k=1}^3\left(\frac{1}{2}(\partial_k A^{jk})(X_t) + A^{jk}(X_t)\frac{\partial_k\psi_\mathrm{p}(X_t)}{\psi_\mathrm{p}(X_t)}\right)dt \tag{3.39}$$

derived from (3.38), where B_t denotes the three-dimensional Brownian motion, and the diffusion term is $\sigma(\mathrm{x})=[A^{jk}]^{\frac{1}{2}}(\mathrm{x})$. This is of course a formal description, since the regularity of ψ_p is not clear at all, and it is thus hopeless to solve (3.39) directly. Instead of this direct approach we use another strategy to construct the diffusion process X associated with L. This is done in Appendix A.

We summarize the properties of X_t in Proposition 3.12 below. Let $\mathscr{X} = C(\mathbb{R};\mathbb{R}^3)$. $X\stackrel{\mathrm{d}}{=}Y$ means that X and Y has the same distribution.

Proposition 3.12. (Diffusion process associated with e^{-tL}) *Let*

$$X_t(\omega)=\omega(t),\quad \omega(\cdot)\in\mathscr{X},$$

be the coordinate mapping process on $(\mathscr{X},B(\mathscr{X}))$, where $B(\mathscr{X})$ denotes the $\sigma-$field generated by cylinder sets. Assume (E1), (E2), (E3), (E5). Then there exists for all $\mathrm{x}\in\mathbb{R}^3$ a probability measure P^x on $(\mathscr{X},B(\mathscr{X}))$ satisfying (1)-(5) below:

- **(1): (Initial distribution)** $\mathrm{P}^\mathrm{x}(X_0=\mathrm{x})=1$.
- **(2): (Continuity)** *$t\mapsto X_t$ is continuous.*
- **(3): (Reflection symmetry)** *$(X_t)_{t\ge 0}$ and $(X_t)_{t\le 0}$ are independent and $X_{-t}\stackrel{\mathrm{d}}{=}X_t$.*
- **(4): (Markov property)** *Let $(\mathcal{F}_t)_{t\ge 0}=\sigma(X_s, 0\le s\le t)$ for $t\ge 0$ and $(\mathcal{F}_t)_{t\le 0}=\sigma(X_s, t\le s\le 0)$ for $t\le 0$ be the associated filtrations. Then $(X_t)_{t\ge 0}$ and $(X_t)_{t\le 0}$ are Markov processes with respect to $(\mathcal{F}_t)_{t\ge 0}$ and $(\mathcal{F}_t)_{t\le 0}$, respectively, i.e.*

$$\mathbb{E}_{\mathrm{P}^\mathrm{x}}[X_{t+s}|\mathcal{F}_s] = \mathbb{E}_{\mathrm{P}^\mathrm{x}}[X_{t+s}|\sigma(X_s)] = \mathbb{E}_{\mathrm{P}^{X_s}}[X_t^{X_s}],$$
$$\mathbb{E}_{\mathrm{P}^\mathrm{x}}[X_{-t-s}|\mathcal{F}_{-s}] = \mathbb{E}_{\mathrm{P}^\mathrm{x}}[X_{-t-s}|\sigma(X_{-s})] = \mathbb{E}_{\mathrm{P}^{X_{-s}}}[X_{-t}^{X_{-s}}]$$

for $s,t\ge 0$, where $\mathbb{E}_{\mathrm{P}^{X_s}}$ means $\mathbb{E}_{\mathrm{P}^\mathrm{y}}$ evaluated at $\mathrm{y}=X_s$.
- **(5): (Shift invariance)**

$$\int d\mu_\mathrm{p}(\mathrm{x})\mathbb{E}_{\mathrm{P}^\mathrm{x}}\left[f_0(X_{t_0})\cdots f_n(X_{t_n})\right] = (f_0|\mathrm{e}^{-(t_1-t_0)L}f_1\cdots\mathrm{e}^{-(t_n-t_{n-1})L}f_n)_{\mathcal{H}_\mathrm{p}} \tag{3.40}$$

for $f_j\in L^\infty(\mathbb{R}^3)$, $j=1,...,n$, and the finite dimensional distribution of X is shift invariant, i.e.:

$$\int d\mu_\mathrm{p}(\mathrm{x})\mathbb{E}_{\mathrm{P}^\mathrm{x}}\left[\prod_{j=1}^n f_j(X_{t_j})\right] = \int d\mu_\mathrm{p}(\mathrm{x})\mathbb{E}_{\mathrm{P}^\mathrm{x}}\left[\prod_{j=1}^n f_j(X_{t_j+s})\right],\quad s\in\mathbb{R},$$

for all bounded Borel measurable functions f_j*,* $j = 1, ..., n$*.*

This proposition may be known, the proof will be however given in Appendix A for self consistency. We define the full probability measure P on $\mathbb{R}^3 \times \mathscr{X}$ by

$$\mathrm{P}(A \times B) = \int_A d\mu_{\mathrm{p}}(\mathrm{x}) \int_B d\mathrm{P}^{\mathrm{x}}.$$

In the sequel we will denote $\mathbb{E}_{\mathrm{P}^{\mathrm{x}}}$ simply by $\mathbb{E}^{\mathrm{x}}$.

4. The Nelson model by path measures

4.1. **Path space approach for boson fields.** Let $\mathcal{X}$ be a real Hilbert space and $a > 0$ a selfadjoint operator on $\mathcal{X}$. It is well known that there exist a probability space (Q, Σ, μ_C) and a linear map:

$$a^{\frac{1}{2}}\mathcal{X} \ni f \mapsto \Phi(f)$$

with values in measurable functions on (Q, Σ) such that

$$\int_Q \mathrm{e}^{\mathrm{i}\Phi(f)} d\mu_C = \mathrm{e}^{-\frac{1}{2}C(f,f)},\ f \in a^{\frac{1}{2}}\mathcal{X},$$

for $C(f, f) = (f|a^{-1}f)_{\mathcal{X}}$. Moreover Σ is generated by the functions $\Phi(f)$, $f \in a^{\frac{1}{2}}\mathcal{X}$.

Such a structure is called the *Gaussian process* indexed by $\mathcal{X}$ with *covariance* C. Let $\mathcal{X}_{\mathbb{C}}$ be the complexification of $\mathcal{X}$. It is well known that $L^2(Q, d\mu_C)$ can be unitarily identified with the bosonic Fock space $\Gamma_{\mathrm{s}}(a^{\frac{1}{2}}\mathcal{X}_{\mathbb{C}})$ by the map

$$U : L^2(Q, d\mu_C) \ni \mathrm{e}^{\mathrm{i}\Phi(f)} \mapsto \mathrm{e}^{\mathrm{i}\phi(f)}\Omega \in \Gamma_{\mathrm{s}}(a^{\frac{1}{2}}\mathcal{X}_{\mathbb{C}}),\quad f \in a^{\frac{1}{2}}\mathcal{X}. \tag{4.1}$$

Here we recall that Ω is the Fock vacuum. If we further identify $\Gamma_{\mathrm{s}}(a^{\frac{1}{2}}\mathcal{X}_{\mathbb{C}})$ with $\Gamma_{\mathrm{s}}(\mathcal{X}_{\mathbb{C}})$ by the map $\Gamma(a^{-\frac{1}{2}})$, we obtain that $\Gamma_{\mathrm{s}}(\mathcal{X}_{\mathbb{C}})$ is unitarily identified with $L^2(Q, d\mu_C)$ by $\mathscr{U}_{\mathrm{f}} = \Gamma(a^{-\frac{1}{2}})U$:

$$\mathscr{U}_{\mathrm{f}} : L^2(Q, d\mu_C) \ni \mathrm{e}^{\mathrm{i}\Phi(f)} \mapsto \mathrm{e}^{\mathrm{i}\phi(a^{-\frac{1}{2}}f)}\Omega \in \Gamma_{\mathrm{s}}(\mathcal{X}_{\mathbb{C}}),\ f \in a^{\frac{1}{2}}\mathcal{X}. \tag{4.2}$$

We will apply this result to $\mathcal{X} = L^2(\mathbb{R}^3)$ and $a = 2\omega$, where ω is defined in (2.8) (note that ω is a real operator). The associated probability space will be denoted by (Q_0, Σ_0, μ_0) and we set

$$\mathcal{H}_{\mathrm{f}} := L^2(Q_0, d\mu_0).$$

Note that under the above identification, any closed operator T on $\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^3))$ affiliated to the abelian von Neumann algebra generated by the $\mathrm{e}^{\mathrm{i}\phi(g)}$ for $g \in L^2_{\mathbb{R}}(\mathbb{R}^3)$ becomes a multiplication operator by a measurable function on (Q_0, Σ_0). We set

$$H_{\mathrm{f}} := \mathscr{U}_{\mathrm{f}}^{-1} d\Gamma(\omega)\mathscr{U}_{\mathrm{f}}.$$

We now recall the well known expression of the semi-group $\mathrm{e}^{-tH_{\mathrm{f}}}$ through Gaussian processes. Let us set $D_t = -\mathrm{i}\partial_t$. Consider the Gaussian process indexed by $L^2_{\mathbb{R}}(\mathbb{R}^4) = L^2_{\mathbb{R}}(\mathbb{R}, dt) \otimes L^2_{\mathbb{R}}(\mathbb{R}^3, dx)$ with covariance

$$C(f, f) = (f|(D_t^2 + \omega^2)^{-1}f)_{L^2(\mathbb{R}^4)} \tag{4.3}$$

and set $a = D_t^2 + \omega^2$. The associated probability space will be denoted by $(Q_{\mathrm{E}}, \Sigma_{\mathrm{E}}, \mu_{\mathrm{E}})$ and the random variables by $\Phi_{\mathrm{E}}(f)$. It is well known that for $t \in \mathbb{R}$, the map

$$j_t : (2\omega)^{\frac{1}{2}}L^2(\mathbb{R}^3) \ni g \mapsto \delta_t \otimes g \in a^{\frac{1}{2}}L^2(\mathbb{R}^4)$$

is isometric, if δ_t denotes the Dirac mass at time t. Moreover one has

$$(\delta_{t_1} \otimes g_1|(D_t^2 + \omega^2)^{-1}\delta_{t_2} \otimes g_2)_{L^2(\mathbb{R}^4)} = (g_1|\frac{1}{2\omega}\mathrm{e}^{-|t_1-t_2|\omega}g_2)_{L^2(\mathbb{R}^3)}. \tag{4.4}$$

For $t \in \mathbb{R}$ and $g \in (2\omega)^{\frac{1}{2}} L^2_{\mathbb{R}}(\mathbb{R}^3)$, we set

$$\Phi_{\rm E}(t,g) := \Phi_{\rm E}(\delta_t \otimes g).$$

It follows that $\Phi_{\rm E}(t,g) \in \cap_{1\leq p<\infty} L^p(Q_{\rm E}, {\rm d}\mu_{\rm E})$. Since the covariance C is invariant under the group τ_s of time translations, we see that

$$T_s := \mathscr{U}_{\rm f}^{-1}\Gamma(\tau_s)\mathscr{U}_{\rm f}, \quad s \in \mathbb{R}$$

is a strongly continuous unitary group on $L^2(Q_{\rm E}, d\mu_{\rm E})$. Clearly

$$T_s\Phi_{\rm E}(t,g) = \Phi_{\rm E}(t+s,g).$$

For $t \in \mathbb{R}$, we denote by $E_t : L^2(Q_{\rm E}, \Sigma_{\rm E}, \mu_{\rm E}) \to L^2(Q_{\rm E}, \Sigma_{\rm E}, \mu_{\rm E})$ the conditional expectation with respect to the σ-algebra Σ_t generated by the $\Phi(t,g)$ for $g \in (2\omega)^{\frac{1}{2}} L^2_{\mathbb{R}}(\mathbb{R}^3)$. As is well known one has $E_t = \mathscr{U}_{\rm f}^{-1}\Gamma(e_t)\mathscr{U}_{\rm f}$ for $e_t = j_t j_t^*$. Clearly $E_0 L^2(Q_{\rm E}, d\mu_{\rm E})$ can be identified with $\mathcal{H}_{\rm f}$ and we will hence consider $\mathcal{H}_{\rm f}$ as a closed subspace of $L^2(Q_{\rm E}, d\mu_{\rm E})$. It follows then from (4.4) that

$$\int_{Q_{\rm E}} \overline{F} T_t G d\mu_{\rm E} = (F|{\rm e}^{-tH_{\rm f}}G)_{\mathcal{H}_{\rm f}},\ t \geq 0, \tag{4.5}$$

for $F, G \in \mathcal{H}_{\rm f}$.

4.2. **Path space representation for the Nelson model.** The Hilbert space $\mathcal{H}_{\rm p} \otimes \mathcal{H}_{\rm f} \cong L^2(\mathbb{R}^3 \times Q_0, d\mu_{\rm p} \otimes d\mu_0)$ and the Hamiltonian $(\mathscr{U}_{\rm p} \otimes \mathscr{U}_{\rm f})H(\mathscr{U}_{\rm p} \otimes \mathscr{U}_{\rm f})^{-1}$ will still be denoted by $\mathcal{H}$ and H respectively. Note that $F \in \mathcal{H}$ can be viewed as a function: $F : \mathbb{R}^3 \ni {\rm x} \mapsto F({\rm x}) \in \mathcal{H}_{\rm f}$ defined almost everywhere. Note also that in this representation the interaction $q\varphi_\rho({\rm x})$ becomes the multiplication operator by the measurable function on $Q_{\rm E} \times \mathbb{R}^3$: $q\Phi_{\rm E}(0, \rho(\cdot - {\rm x}))$.

Theorem 4.1. (1) *Assume (E1), (E2), (E3), (E5). Let $F, G \in \mathcal{H}$. Then for all $t \geq 0$*

$$(F|{\rm e}^{-tH}G)_{\mathcal{H}} = \int d\mu_{\rm p}({\rm x})\mathbb{E}^{\rm x}\left[(F(X_0)|{\rm e}^{-q\int_0^t \Phi_{\rm E}(s,\rho(\cdot - X_s))ds} T_t G(X_t))_{L^2(Q_{\rm E})}\right]. \tag{4.6}$$

(2) *In particular*

$$(\mathbb{1}|{\rm e}^{-TH}\mathbb{1})_{\mathcal{H}} = \int d\mu_{\rm p}({\rm x})\mathbb{E}^{\rm x}\left[{\rm e}^{(q^2/2)\int_0^T dt \int_0^T ds W(X_t, X_s, |t-s|)}\right], \tag{4.7}$$

where

$$W({\rm x}, {\rm y}, |t|) = (\rho(\cdot - {\rm x}), \frac{{\rm e}^{-|t|\omega}}{2\omega}\rho(\cdot - {\rm y}))_{L^2(\mathbb{R}^3)}. \tag{4.8}$$

Proof. Suppose that $G \in L^\infty(\mathbb{R}^3 \times Q_0, d\mu_{\rm p} \otimes d\mu_0)$. By the Trotter-Kato product formula [KM] we have

$${\rm e}^{-tH} = s - \lim_{n\to\infty}\left({\rm e}^{-(t/n)L}{\rm e}^{-(t/n)\Phi_{\rm E}(0, q\rho(\cdot - {\rm x}))}{\rm e}^{-(t/n)H_{\rm f}}\right)^n.$$

Using the factorization formula (4.5), the Markov property of E_t, we have

$$\begin{aligned}(F|{\rm e}^{-tH}G) \\ = \lim_{n\to\infty}\int d\mu_{\rm p}({\rm x})\mathbb{E}^{\rm x}\left[\left(F(X_0), {\rm e}^{-\sum_{j=0}^n \frac{t}{n}\Phi_{\rm E}(jt/n, q\rho(\cdot - X_{jt/n}))} T_t G(X_t)\right)\right].\end{aligned} \tag{4.9}$$

We set $\rho_s = \rho(\cdot - X_s)$ for $s \in \mathbb{R}$. Using that $s \mapsto X_s \in \mathbb{R}^3$ is continuous, we see that

$$\mathbb{R} \ni s \mapsto \rho_s \in \omega^{\frac{1}{2}} L^2(\mathbb{R}^3) \tag{4.10}$$

is continuous. This implies that

$$\mathbb{R} \ni s \mapsto \Phi_{\rm E}(s, \rho_s) \in \cap_{1\leq p<\infty} L^p(Q_{\rm E}, d\mu_{\rm E}) \tag{4.11}$$

is also continuous. In fact

$$\Phi_{\mathrm{E}}(t,\rho_t)-\Phi_{\mathrm{E}}(s,\rho_s)=\Phi_{\mathrm{E}}(t,\rho_t)-\Phi_{\mathrm{E}}(s,\rho_t)+\Phi_{\mathrm{E}}(s,\rho_t-\rho_s).$$

The first term in the right hand side tends to 0 when $s\to t$ in $\cap_{1\le p<\infty}L^p(Q_{\mathrm{E}},d\mu_{\mathrm{E}})$ since the time translation group $\{T_t\}_{t\in\mathbb{R}}$ is strongly continuous on $\cap_{1\le p<\infty}L^p(Q_{\mathrm{E}},d\mu_{\mathrm{E}})$. The same is true for the second term using (4.10). It follows from (4.11) that

$$\sum_{j=0}^{n}\frac{t}{n}\Phi_{\mathrm{E}}(jt/n,\rho(\cdot-X_{jt/n}))\to\int_0^t\Phi_{\mathrm{E}}(s,\rho(\cdot-X_s))ds$$

in $\cap_{1\le p<\infty}L^p(Q_{\mathrm{E}})$, when $n\to\infty$. We claim now that

(4.12)
$$\lim_{n\to+\infty}\exp\left(-\sum_{j=0}^{n}\frac{t}{n}\Phi_{\mathrm{E}}(jt/n,q\rho(\cdot-X_{jt/n}))\right)=\exp\left(-q\int_0^t\Phi_{\mathrm{E}}(s,\rho(\cdot-X_s))ds\right)$$

in $\cap_{1\le p<\infty}L^p(Q_{\mathrm{E}})$. To prove (4.12) we set

$$f_n=(t/n)\sum_{j=1}^{n}\delta_{tj/n}\otimes\rho(\cdot-X_{tj/n}(\omega)),$$
$$f=\int_0^t\delta_s\otimes\rho(\cdot-X_s)ds.$$

The map

$$\mathbb{R}\ni s\mapsto\delta_s\otimes\rho(\cdot-X_s)\in(D_t^2+\omega^2)^{\frac12}L^2(\mathbb{R}^4)$$

is continuous. Note that

$$\Phi_{\mathrm{E}}(f_n)=\sum_{j=0}^{n}\frac{t}{n}\Phi_{\mathrm{E}}(jt/n,\rho(\cdot-X_{jt/n}))$$
$$\Phi_{\mathrm{E}}(f)=\int_0^t\Phi_{\mathrm{E}}(s,\rho(\cdot-X_s))ds.$$

We write now

$$\mathrm{e}^{-q\Phi_{\mathrm{E}}(f)}-\mathrm{e}^{-q\Phi_{\mathrm{E}}(f_n)}=q\int_0^1\mathrm{e}^{-q\Phi_{\mathrm{E}}(rf+(1-r)f_n)}\Phi_{\mathrm{E}}(f_n-f)dr.$$

We use the fact that $\Phi_{\mathrm{E}}(g)$ is a Gaussian random variable and hence

(4.13)
$$\|\mathrm{e}^{\Phi_{\mathrm{E}}(g)}\|_{L^p}^p=\mathrm{e}^{p^2C(g,g)/2}=\mathrm{e}^{(p^2/2)\|\Phi_{\mathrm{E}}(g)\|_2^2},\ 1\le p<\infty,\ g\in(D_t^2+\omega^2)^{\frac12}L^2(\mathbb{R}^4).$$

Applying Hölder's inequality we obtain (4.12). To complete the proof of (1) it remains to justify the exchange of limit and integral. To do this we note that the family of functions

$$F(X_0)\mathrm{e}^{-q\Phi_{\mathrm{E}}(f_n)}T_tG(X_t),\ n\in\mathbb{N}$$

is equi-integrable if $F\in L^\infty$, $G\in\mathcal{H}$, since it is uniformly bounded in L^p for some $p>1$, by Hölder's inequality and (4.12). This completes the proof of (1) for $G\in L^\infty$ and $F\in\mathcal{H}$.

Next suppose that $G,F\in\mathcal{H}$. We can suppose $F,G\ge0$ without loss of generality. Let $G_n=\min\{G,n\}$, $n\in\mathbb{N}$. Thus $(F|\mathrm{e}^{-tH}G_n)\to(F|\mathrm{e}^{-tH}G)$ as $n\to\infty$ and $F(X_0)\mathrm{e}^{-q\int_0^t\Phi_{\mathrm{E}}(s,\rho(\cdot-X_s))ds}T_tG_n(X_t)$ is monotonously increasing as $N\uparrow\infty$. By the monotone convergence theorem we get that

$$F(X_0)\mathrm{e}^{-q\int_0^t\Phi_{\mathrm{E}}(s,\rho(\cdot-X_s))ds}T_tG(X_t)\in L^1(\mathbb{R}^3\times\mathscr{X}\times Q_{\mathrm{E}},d\mathrm{P}\otimes d\mu_{\mathrm{E}})$$

and (1) follows. Applying (1) to $F = G = \mathbb{1}$, we get

$$(\mathbb{1}|\mathrm{e}^{-tH}\mathbb{1}) = \int d\mu_{\mathrm{p}}(\mathrm{x})\mathbb{E}^{\mathrm{x}}\left[(\mathbb{1}, \mathrm{e}^{q\Phi_{\mathrm{E}}(f)}\mathbb{1})\right] = \int d\mu_{\mathrm{p}}(\mathrm{x})\mathbb{E}^{\mathrm{x}}\left[\mathrm{e}^{(q^2/2)C(f,f)}\right].$$

Using (4.3), we get

$$\begin{aligned} C(f,f) &= \int_0^T dt \int_0^T ds(\delta_t \otimes \rho(\cdot - X_t)|(D_t^2 + \omega^2)^{-1}\delta_s \otimes \rho(\cdot - X_s)) \\ &= \int_0^T dt \int_0^T ds W(X_t, X_s, |t-s|). \end{aligned}$$

This completes the proof of the theorem. □

Proposition 4.2. *Assume (E1), (E2), (E3), (E5). Then* e^{-tH} *is positivity improving for all* $t > 0$.

Proof. Let $t > 0$ and $F, G \in \mathcal{H}$ with $F, G \geq 0$, $F, G \neq 0$. We need to prove that $(F|\mathrm{e}^{-tH}G) > 0$. Since $\int_0^t \Phi_{\mathrm{E}}(s, \rho(\cdot - X_s))ds$ belongs to L^1, $\mathrm{e}^{-\int_0^t \Phi_{\mathrm{E}}(s,\rho(\cdot - X_s))ds} > 0$ a.e. Therefore it suffices to prove that

$$\int d\mu_{\mathrm{p}}(\mathrm{x})\mathbb{E}^{\mathrm{x}}\left[(F(X_0)|T_t G(X_t))\right] = (F|\mathrm{e}^{-tH_0}G) > 0. \tag{4.14}$$

The equality above immediately shows that e^{-tH_0} is positivity preserving for all $t > 0$. Moreover $\mathbb{1} \otimes \mathbb{1}$ is the unique strictly positive ground state of H_0. Therefore by [RS, Theorem XIII.44] e^{-tH_0} is positivity improving for all $t > 0$ and hence (4.14) holds. This completes the proof of the proposition. □

We complete this section by stating a standard abstract criterion for the existence of a ground state for generators of positivity improving heat semi-groups.

Lemma 4.3. *Let* (Q, Σ, μ) *be a probability space, and* H *a bounded below selfadjoint operator on* $L^2(Q, \Sigma, \mu)$ *such that* e^{-tH} *is positivity improving for* $t > 0$. *Set*

$$\gamma(T) := \frac{(\mathbb{1}|\mathrm{e}^{-TH}\mathbb{1})^2}{\|\mathrm{e}^{-TH}\mathbb{1}\|^2},$$

and $E = \inf \sigma(H)$. *Then* $\lim_{T\to+\infty} \gamma(T) = \|\mathbb{1}_{\{E\}}(H)\mathbb{1}\|^2$. *In particular* H *has a ground state iff* $\lim_{T\to+\infty} \gamma(T) \neq 0$.

Note that by Proposition 4.2, Lemma 4.3 can be applied to the Nelson Hamiltonian H.

Proof. We can assume that $E = 0$, so that $\text{s-}\lim_{T\to+\infty} \mathrm{e}^{-TH} = \mathbb{1}_{\{0\}}(H)$. If 0 is an eigenvalue, then by Perron-Frobenius arguments, $\mathbb{1}_{\{0\}}(H) = |u)(u|$ for some $u > 0$. It follows that $\lim_{T\to+\infty} \gamma(T) = (u|1)^2$. Assume now that H has no ground state and that there exists a sequence $T_n \to +\infty$ such that $\gamma(T_n) \geq \delta^2 > 0$. This implies that $(\mathbb{1}|\mathrm{e}^{-T_nH}\mathbb{1}) \geq \delta(\mathbb{1}|\mathrm{e}^{-2T_nH}\mathbb{1})^{\frac{1}{2}}$. Letting $n \to +\infty$, we obtain that $\|\mathbb{1}_{\{0\}}(H)\mathbb{1}\| \geq \delta$, which is a contradiction. □

5. Absence of ground state

5.1. **Proof of Theorem 2.3.** In this section we assume the hypotheses of Theorem 2.3. We first prove some upper and lower bounds on the interaction kernels $W(\mathrm{x}, \mathrm{y}, t)$. This is the only place where the hypotheses (B2) on fast decay of the variable mass $m(x)$ and (B3) on the positivity of the space cutoff function ρ enter.

Set $\rho_{\mathrm{x}}(x) = \rho(x - \mathrm{x})$. We recall from (4.8) that

$$W(\mathrm{x}, \mathrm{y}, t) = (\rho_{\mathrm{x}}|\frac{\mathrm{e}^{-t\omega}}{2\omega}\rho_{\mathrm{y}}), \ \mathrm{x}, \mathrm{y} \in \mathbb{R}^3, \ t \geq 0.$$

We set $h_\infty = -\Delta_x$, $\omega_\infty = h_\infty^{\frac{1}{2}}$, and denote by $W_\infty(\mathrm{x},\mathrm{y},t)$ the analog potential for ω replaced by ω_∞. Note also that

$$\mathrm{e}^{-t\omega_\infty}(x,y) = \frac{1}{\pi^2}\frac{t}{(|x-y|^2+|t|^2)^2},$$

which using the identity $\frac{1}{\lambda}\mathrm{e}^{-t\lambda} = \int_t^{+\infty}\mathrm{e}^{-s\lambda}ds$ yields

$$\begin{aligned} W_\infty(\mathrm{x},\mathrm{y},t) &= \frac{1}{4\pi^2}\int\frac{\rho(x-\mathrm{x})\rho(y-\mathrm{y})}{|x-y|^2+t^2}dxdy \\ &= \frac{1}{4\pi^2}\int\frac{\rho(x)\rho(y)}{|x-y+\mathrm{x}-\mathrm{y}|^2+t^2}dxdy. \end{aligned} \tag{5.1}$$

We also have

$$W_\infty(\mathrm{x},\mathrm{y},|t|) = \frac{1}{2}\int\frac{|\hat\rho|^2(k)\mathrm{e}^{-ik\cdot(\mathrm{x}-\mathrm{y})}}{|k|}\mathrm{e}^{-|t||k|}\,dk. \tag{5.2}$$

Lemma 5.1. (1) *$W(\mathrm{x},\mathrm{y},|t|) \geq 0$ and $W_\infty(\mathrm{x},\mathrm{y},|t|) \geq 0$,*
(2) *Assume (B2). Then there exist constants $C_j > 0$, $j = 1,2,3,4$, such that*

$$C_1 W_\infty(\mathrm{x},\mathrm{y},C_2|t|) \leq W(\mathrm{x},\mathrm{y},|t|) \leq C_3 W_\infty(\mathrm{x},\mathrm{y},C_4|t|) \tag{5.3}$$

for all $\mathrm{x},\mathrm{y}\in\mathbb{R}^3$ and $t\in\mathbb{R}$.

Proof. We note that the function $f(\lambda) = \mathrm{e}^{-\sqrt{\lambda}}$ on $[0,\infty)$ is completely monotone, i.e., $(-1)^n df(\lambda)/d\lambda^n \geq 0$ and that $f(+0) = 0$. Then by Bernstein's theorem [BF] there exists a Borel probability measure m on $[0,\infty)$ such that

$$\mathrm{e}^{-\sqrt{\lambda}} = \int_0^\infty \mathrm{e}^{-s\lambda}dm(s),$$

and actually $dm(s) = \dfrac{1}{2\sqrt{\pi}}\dfrac{\mathrm{e}^{-1/(4s)}}{s^{3/2}}ds$. Hence

$$\mathrm{e}^{-t\omega} = \int_0^\infty \mathrm{e}^{-st^2\omega^2}dm(s) = \frac{1}{2\sqrt{\pi}}\int_0^\infty\frac{t\mathrm{e}^{-t^2/(4s)}}{s^{3/2}}\mathrm{e}^{-s\omega^2}ds.$$

It follows that

$$W(\mathrm{x},\mathrm{y},|t|) = \frac{1}{2}\int_{|t|}^\infty dr(\rho_\mathrm{x}|\mathrm{e}^{-r\omega}\rho_\mathrm{y}) = \frac{1}{4\sqrt{\pi}}\int_{|t|}^\infty dr\int_0^\infty\frac{r\mathrm{e}^{-r^2/(4p)}}{p^{3/2}}(\rho_\mathrm{x}|\mathrm{e}^{-ph}\rho_\mathrm{y})dp.$$

This implies (1) since e^{-ph} is positivity preserving.

To prove (2), we note that by Proposition 3.2 there exist constants c_j such that

$$c_1\mathrm{e}^{c_2t\Delta}(x,y) \leq \mathrm{e}^{-th}(x,y) \leq c_3\mathrm{e}^{c_4t\Delta}(x,y). \tag{5.4}$$

Since ρ_x and ρ_y are non-negative, we see that by change of variables that ,

$$c_1c_2W_\infty(\mathrm{x},\mathrm{y},\sqrt{c_2}|t|) \leq W(\mathrm{x},\mathrm{y},|t|) \leq c_3c_4W_\infty(\mathrm{x},\mathrm{y},\sqrt{c_4}|t|),$$

which completes the proof of the lemma. □

Let μ_T be the probability measure on $\mathbb{R}^3\times\mathscr{X}$ being absolutely continuous with respect to P such that

$$d\mu_T = \frac{1}{Z_T}\mathrm{e}^{(q^2/2)\int_{-T}^T ds\int_{-T}^T dtW(X_s,X_t,|s-t|)}d\mathrm{P}, \tag{5.5}$$

where Z_T denotes the normalizing constant such that μ_T becomes a probability measure.

Lemma 5.2. *One has*

$$\gamma(T) \leq \mathbb{E}_{\mu_T}\left[\mathrm{e}^{-q^2\int_{-T}^0 ds\int_0^T dtW(X_s,X_t,|s-t|)}\right]. \tag{5.6}$$

Proof. Using Theorem 4.1 (2) and the shift invariance of X_t (see Proposition 3.12) it follows that the denominator of $\gamma(T)$ equals

$$\|\mathrm{e}^{-TH}\mathbb{1}\|^2 = (\mathbb{1}|\mathrm{e}^{-2TH}\mathbb{1}) = \int d\mu_{\mathrm{p}}(\mathrm{x})\mathbb{E}^{\mathrm{x}}\left[\mathrm{e}^{(q^2/2)\int_{-T}^{T}dt\int_{-T}^{T}dsW(X_t,X_s,|t-s|)}\right] = Z_T.$$

The numerator of $\gamma(T)$ can be estimated by the Cauchy-Schwarz inequality and shift invariance of X_t:

$$\begin{aligned}(\mathbb{1}|\mathrm{e}^{-tH}\mathbb{1})^2 = & \left(\int d\mu_{\mathrm{p}}(\mathrm{x})\mathbb{E}^{\mathrm{x}}\left[\mathrm{e}^{(q^2/2)\int_0^T dt\int_0^T dsW}\right]\right)^2 \\ \leq & \int d\mu_{\mathrm{p}}(\mathrm{x})\left(\mathbb{E}^{\mathrm{x}}\left[\mathrm{e}^{(q^2/2)\int_0^T dt\int_0^T dsW}\right]\right)\left(\mathbb{E}^{\mathrm{x}}\left[\mathrm{e}^{(q^2/2)\int_{-T}^0 dt\int_{-T}^0 dsW}\right]\right) \\ = & \int d\mu_{\mathrm{p}}(\mathrm{x})\mathbb{E}^{\mathrm{x}}\left[\mathrm{e}^{(q^2/2)\left(\int_0^T dt\int_0^T dsW+\int_{-T}^0 dt\int_{-T}^0 dsW\right)}\right],\end{aligned}$$

where in the last line we use the fact that X_s and X_t are independent for $s \leq 0 \leq t$. Next we note that if $F(s,t) = F(t,s)$ we have

$$\begin{aligned}&\int_0^T ds\int_0^T dtF(s,t) + \int_{-T}^0 ds\int_{-T}^0 dtF(s,t) \\ &= \int_{-T}^T ds\int_{-T}^T dtF(s,t) - 2\int_{-T}^0 ds\int_0^T dtF(s,t).\end{aligned}$$

We can apply this identity to $F(s,t) = W(X_s, X_t, |t-s|)$ and obtain

$$\begin{aligned}&\left(\int d\mu_{\mathrm{p}}(\mathrm{x})\mathbb{E}^{\mathrm{x}}\left[\mathrm{e}^{(q^2/2)\int_0^T dt\int_0^T dsW}\right]\right)^2 \\ &\leq \int d\mu_{\mathrm{p}}(\mathrm{x})\mathbb{E}^{\mathrm{x}}\left[\mathrm{e}^{-q^2\int_{-T}^0 ds\int_0^T dtW+(q^2/2)\int_{-T}^T ds\int_{-T}^T dtW}\right],\end{aligned}$$

which using the definition of μ_T completes the proof of the lemma. □

Let us take λ such that

$$\frac{1}{\delta+1} < \lambda < 1, \tag{5.7}$$

where δ is the exponent in Assumption (E5) and set

$$A_T := \left\{(\mathrm{x},\omega) \in \mathbb{R}^3 \times \mathscr{X} \mid \sup_{|s|\leq T}|X_s(\omega)| \leq T^{\lambda}\right\}. \tag{5.8}$$

The proof of Theorem 2.3 will follow immediately from the following two lemmas.

Lemma 5.3. *One has*

$$\lim_{T\to\infty}\mathbb{E}_{\mu_T}\left[\mathbb{1}_{A_T}\mathrm{e}^{-q^2\int_{-T}^0 ds\int_0^T dtW(X_s,X_t,|s-t|)}\right] = 0.$$

Lemma 5.4. *One has*

$$\lim_{T\to\infty}\mathbb{E}_{\mu_T}\left[\mathbb{1}_{A_T^c}\mathrm{e}^{-q^2\int_{-T}^0 ds\int_0^T dtW(X_s,X_t,|s-t|)}\right] = 0.$$

Proof of Theorem 2.3. By Lemmas 5.3, 5.4 and 5.2 it follows that $\lim_{T\to+\infty}\gamma(T) = 0$. We apply then Lemma 4.3. □

5.2. **Proofs of Lemmas 5.3 and 5.4.** We prove in this section Lemmas 5.3 and 5.4.

Proof of Lemma 5.3. By Lemma 5.1, it suffices to prove that

$$\lim_{T\to\infty}\mathbb{E}_{\mu_T}\left[\mathbb{1}_{A_T}\mathrm{e}^{-C_1\int_{-T}^0 ds\int_0^T dtW_\infty(X_s,X_t,C_2|s-t|)}\right] = 0. \tag{5.9}$$

The proof is similar to [LMS]. Let

$$\begin{aligned}\Delta_T &= \{(s,t)|0 \leq s \leq T, 0 \leq t \leq T, 0 \leq s+t \leq T/\sqrt{2}\}, \\ \Delta_T' &= \{(s,t)|0 \leq s \leq T/\sqrt{2}, -s \leq t \leq s\},\end{aligned}$$

so that

$$
\begin{aligned}
&\textstyle\int_{-T}^{0} dt \int_0^T dt \frac{1}{a^2+|t-s|^2}\\
&\geq \textstyle\int\int_{\Delta_T} dsdt \frac{1}{a^2+|s+t|^2}\\
&= \textstyle\int\int_{\Delta'_T} dsdt \frac{1}{a^2+s^2} = \log\left(\frac{a^2+T^2/2}{a^2}\right).
\end{aligned}
\tag{5.10}
$$

We note now that $|x-y+\mathrm{x}-\mathrm{y}|^2+|t-s|^2 \leq 8T^{2\lambda}+2|x-y|^2+|t-s|^2$ uniformly for $|\mathrm{x}| \leq T^\lambda$, $|\mathrm{y}| \leq T^\lambda$. Using (5.1) and (5.10) this yields

$$
\begin{aligned}
&\mathbb{1}_{A_T} \textstyle\int_{-T}^{0} ds \int_0^T dt W_\infty(X_s, X_t, C_2|s-t|)\\
&\geq \tfrac{1}{4\pi^2}\mathbb{1}_{A_T} \textstyle\int_{-T}^{0} ds \int_0^T dt \int dxdy \frac{\rho(x)\rho(y)}{8T^{2\lambda}+2|x-y|^2+C_2|t-s|^2}\\
&\geq \tfrac{1}{4C_2\pi^2}\mathbb{1}_{A_T} \textstyle\int dxdy \rho(x)\rho(y) \log\left(\frac{8T^{2\lambda}+2|x-y|^2+C_2T^2/2}{8T^{2\lambda}+2|x-y|^2}\right).
\end{aligned}
$$

Note that $\rho \geq 0$ and $\lambda < 1$. Since the right-hand side above goes to $+\infty$ as $T \to \infty$, (5.9) follows. □

Proof of Lemma 5.4.
Using again Lemma 5.1 it suffices to prove

$$
\lim_{T\to\infty} \mathbb{E}_{\mu_T}\left[\mathbb{1}_{A_T^c} \mathrm{e}^{-C_1 \int_{-T}^0 ds \int_0^T dt W_\infty(X_s,X_t,C_2|s-t|)}\right] = 0. \tag{5.11}
$$

By a change of variables we see that

$$
\int_{-T}^{T}\int_{-T}^{T} \mathrm{e}^{-|s-t|\lambda} dsdt \leq \int_{-\sqrt{2}T}^{\sqrt{2}T}\int_{-\sqrt{2}T}^{\sqrt{2}T} \mathrm{e}^{-|t|\lambda} dsdt \leq CT\lambda^{-1}, \ \forall \lambda > 0.
$$

Using (5.2) and Lemma 5.1 this implies that

$$
0 \leq \int_{-T}^{T} ds \int_{-T}^{T} dt W_\infty(X_s, X_t, C_2|s-t|) \leq C\frac{T}{2}\|\hat\rho/|k|\|^2, \tag{5.12}
$$

$$
0 \leq \int_{-T}^{T} ds \int_{-T}^{T} dt W(X_s, X_t, C_2|s-t|) \leq C\frac{T}{2}\|\hat\rho/|k|\|^2. \tag{5.13}
$$

Set $\frac{1}{2}\|\hat\rho/|k|\|^2 = \xi$. Hence (5.12), (5.13) and the Cauchy-Schwartz inequality yield that

$$
\begin{aligned}
&\mathbb{E}_{\mu_T}\left[\mathbb{1}_{A_T^c}\mathrm{e}^{-\int_{-T}^0 ds \int_0^T dt W_\infty}\right] \leq \mathrm{e}^{TC\xi}\mathbb{E}_{\mu_T}\left[\mathbb{1}_{A_T^c}\right]\\
&= \mathrm{e}^{TC\xi}\frac{\int \mathbb{1}_{A_T^c}\mathrm{e}^{(q^2/2)\int_{-T}^T ds \int_{-T}^T dt W} d\mathrm{P}}{\int \mathrm{e}^{(q^2/2)\int_{-T}^T ds \int_{-T}^T dt W} d\mathrm{P}}\\
&\leq \mathrm{e}^{TC\xi}\frac{\left(\int \mathrm{e}^{q^2\int_{-T}^T ds \int_{-T}^T dt W} d\mathrm{P}\right)^{1/2}}{\int \mathrm{e}^{(q^2/2)\int_{-T}^T ds \int_{-T}^T dt W} d\mathrm{P}}\left(\int \mathbb{1}_{A_T^c} d\mathrm{P}\right)^{1/2} \leq \mathrm{e}^{TC'\xi}\left(\int \mathbb{1}_{A_T^c} d\mathrm{P}\right)^{1/2}.
\end{aligned}
$$

By Lemma 5.5 below we know that there exist constants $a, b > 0$ such that

$$
\int \mathbb{1}_{A_T^c} d\mathrm{P} \leq T^{-\lambda}(a+bT)^{\frac{1}{2}}\mathrm{e}^{-T^{\lambda(\delta+1)}}. \tag{5.14}
$$

Since $\lambda(\delta+1) > 1$ this completes the proof of the lemma. □

Lemma 5.5. *There exist constants $a, b > 0$ such that (5.14) is satisfied, where $\delta > 0$ is the exponent appearing in Assumption (E5).*

5.3. **Proof of Lemma 5.5.** This section is devoted to the proof of Lemma 5.5. Let $G \subset \mathbb{R}^3$ be a closed set, and $T > 0$ and $n \in \mathbb{N}$ are fixed. We define the stopping time

$$\tau := \inf\{T_j \mid j = 0, 1, ..., n, X_{T_j} \in G\}, \quad T_j = \frac{j}{n}T. \tag{5.15}$$

Lemma 5.6. *Let $\psi \in \mathcal{H}_{\rm p}$ with $\psi \geq 0$ and $\psi \geq 1$ on G. Let τ be in (5.15). Then for all $0 < \varrho < 1$ one has*

$$\int d\mu_{\rm p}({\rm x})(\mathbb{E}^{\rm x}[\varrho^\tau])^2 \leq (\psi|\psi) + \frac{\varrho^{T/n}}{1-\varrho^{T/n}}(\psi|(\mathbb{1} - {\rm e}^{-(T/n)L})\psi).$$

Proof. Set $\psi_\varrho({\rm x}) = \mathbb{E}^{\rm x}[\varrho^\tau]$. By the definition of τ we can see that

$$\psi_\varrho({\rm x}) = 1, \quad {\rm x} \in G, \tag{5.16}$$

since $\tau = 0$ in the case X_s starts from the inside of G. We can directly see that

$${\rm e}^{-sL}\psi_\varrho({\rm x}) = \mathbb{E}^{\rm x}[\mathbb{E}^{X_s}[\varrho^\tau]] = \mathbb{E}^{\rm x}[\varrho^{\tau\circ\theta_s}]$$

by the Markov property, where θ_s is the shift on $\mathscr{X}$ defined by $(\theta_s\omega)(t) = \omega(t+s)$ for $\omega \in \mathscr{X}$. Note that $(\tau \circ \theta_{T/n})(\omega) = \tau(\omega) - T/n \geq 0$, when ${\rm x} = X_0(\omega) \notin G$. Hence

$$\varrho^{T/n}{\rm e}^{-(T/n)L}\psi_\varrho({\rm x}) = \psi_\varrho({\rm x}), \quad {\rm x} \in G^c. \tag{5.17}$$

Clearly

$$\int d\mu_{\rm p}({\rm x})(\mathbb{E}^{\rm x}[\varrho^\tau])^2 = (\psi_\varrho|\psi_\varrho) \leq (\psi_\varrho|\psi_\varrho) + \frac{\varrho^{T/n}}{1-\varrho^{T/n}}(\psi_\varrho|(\mathbb{1} - {\rm e}^{-(T/n)L})\psi_\varrho).$$

By (5.17) the right-hand side above equals

$$(\mathbb{1}_G\psi_\varrho|\mathbb{1}_G\psi_\varrho) + \frac{\varrho^{T/n}}{1-\varrho^{T/n}}(\mathbb{1}_G\psi_\varrho|(\mathbb{1} - {\rm e}^{-(T/n)L})\psi_\varrho). \tag{5.18}$$

Next

$$\begin{aligned}(\mathbb{1}_G\psi_\varrho|(\mathbb{1} - {\rm e}^{-(T/n)L})\psi_\varrho) &= (\mathbb{1}_G\psi_\varrho|(\mathbb{1} - {\rm e}^{-(T/n)L})\mathbb{1}_G\psi_\varrho) + (\mathbb{1}_G\psi_\varrho|(\mathbb{1} - {\rm e}^{-(T/n)L})\mathbb{1}_{G^c}\psi_\varrho)\\ &= (\mathbb{1}_G\psi_\varrho|(\mathbb{1} - {\rm e}^{-(T/n)L})\mathbb{1}_G\psi_\varrho) - (\mathbb{1}_G\psi_\varrho|{\rm e}^{-(T/n)L}\mathbb{1}_{G^c}\psi_\varrho)\\ &\leq (\mathbb{1}_G\psi_\varrho|(\mathbb{1} - {\rm e}^{-(T/n)L})\mathbb{1}_G\psi_\varrho),\end{aligned}$$

since ${\rm e}^{-sL}$ has a positive kernel. Hence

$$\int d\mu_{\rm p}({\rm x})(\mathbb{E}^{\rm x}[\varrho^\tau])^2 \leq (\psi_\varrho\mathbb{1}_G|\psi_\varrho\mathbb{1}_G) + \frac{\varrho^{T/n}}{1-\varrho^{T/n}}(\psi_\varrho\mathbb{1}_G|(\mathbb{1} - {\rm e}^{-(T/n)L})\psi_\varrho\mathbb{1}_G). \tag{5.19}$$

Note that $\psi_\varrho({\rm x})\mathbb{1}_G({\rm x}) \leq \psi({\rm x})$ for all ${\rm x} \in \mathbb{R}^3$. Then

$$\begin{aligned}&(\psi_\varrho\mathbb{1}_G|\psi_\varrho\mathbb{1}_G) + \frac{\varrho^{T/n}}{1-\varrho^{T/n}}(\psi_\varrho\mathbb{1}_G|(\mathbb{1} - {\rm e}^{-(T/n)L})\psi_\varrho\mathbb{1}_G)\\ &\leq (\psi|\psi) + \frac{\varrho^{T/n}}{1-\varrho^{T/n}}(\psi|(\mathbb{1} - {\rm e}^{-(T/n)L})\psi).\end{aligned} \tag{5.20}$$

Then combining (5.19) and (5.20) we prove the lemma. □

Proposition 5.7. *Let $\Lambda > 0$ and $f \in C(\mathbb{R}^3) \cap D(L^{1/2})$. Then*

$${\rm P}\left(\sup_{0\leq s\leq T}|f(X_s)| \geq \Lambda\right) \leq \frac{\rm e}{\Lambda}\sqrt{(f|f) + T(L^{1/2}f|L^{1/2}f)}. \tag{5.21}$$

Proof. The proof is a modification of [KV, Lemma 1.4 and Theorem 1.12]. We fix $T > 0$ and $n \in \mathbb{N}$ and define the stopping time τ as in (5.15) for the closed set $G := \{x \in \mathbb{R}^3 \mid |f(x)| \geq \Lambda\}$. It follows that

$$\mathrm{P}\left(\sup_{j=0,\ldots,n} |f(X_{T_j})| \geq \Lambda\right) = \mathrm{P}(\tau \leq T).$$

Let $0 < \varrho < 1$ which will be chosen later. Clearly

$$\mathrm{P}(\tau \leq T) \leq \int \varrho^{\tau - T} d\mathrm{P} \leq \varrho^{-T} \int \varrho^{\tau} d\mathrm{P} \leq \varrho^{-T} \left(\int d\mu_{\mathrm{p}}(\mathrm{x}) (\mathbb{E}^{\mathrm{x}}[\varrho^{\tau}])^2\right)^{1/2}. \tag{5.22}$$

Let $\psi \in \mathcal{H}_{\mathrm{p}}$ be any function such that $\psi \geq 0$ and $\psi(x) \geq 1$ on G. Then applying Lemma 5.6 we get

$$\int d\mu_{\mathrm{p}}(\mathrm{x}) (\mathbb{E}^{\mathrm{x}}[\varrho^{\tau}])^2 \leq (\psi|\psi) + \frac{\varrho^{T/n}}{1 - \varrho^{T/n}} (\psi|(\mathbb{1} - \mathrm{e}^{-(T/n)L})\psi). \tag{5.23}$$

Since $|f(x)| \geq \Lambda$ on G we can put $\psi = |f(x)|/\Lambda$ in (5.23) and get

$$\int d\mu_{\mathrm{p}}(\mathrm{x}) (\mathbb{E}^{\mathrm{x}}[\varrho^{\tau}])^2 \leq \frac{1}{\Lambda^2}(f|f) + \frac{\varrho^{T/n}}{1 - \varrho^{T/n}} \frac{1}{\Lambda^2} (|f||(\mathbb{1} - \mathrm{e}^{-(T/n)L})|f|). \tag{5.24}$$

Since $(|f||(\mathbb{1} - \mathrm{e}^{-(T/n)L})|f|). \leq (f|(\mathbb{1} - \mathrm{e}^{-(T/n)L})f)$, we have

$$\int d\mu_{\mathrm{p}}(\mathrm{x}) (\mathbb{E}^{\mathrm{x}}[\varrho^{\tau}])^2 \leq \frac{1}{\Lambda^2}(f|f) + \frac{\varrho^{T/n}}{1 - \varrho^{T/n}} \frac{1}{\Lambda^2} (f|(\mathbb{1} - \mathrm{e}^{-(T/n)L})f). \tag{5.25}$$

Then by (5.22),

$$\mathrm{P}\left(\sup_{j=0,\ldots,n} |f(X_{T_j})| \geq \Lambda\right) \leq \frac{\varrho^{-T}}{\Lambda} \left((f|f) + \frac{\varrho^{T/n}}{1 - \varrho^{T/n}} (f|(\mathbb{1} - \mathrm{e}^{-(T/n)L})f)\right)^{\frac{1}{2}}.$$

Set $\varrho = \mathrm{e}^{-1/T}$. Then since $\frac{\varrho^{T/n}}{1 - \varrho^{T/n}} \leq n$

$$\mathrm{P}\left(\sup_{j=0,\ldots,n} |f(X_{T_j})| \geq \Lambda\right) \leq \frac{\mathrm{e}}{\Lambda} \left((f|f) + n(f|(\mathbb{1} - \mathrm{e}^{-(T/n)L})f)\right)^{\frac{1}{2}} \tag{5.26}$$

follows. Since $(f|(\mathbb{1} - \mathrm{e}^{-(T/n)L})f) \leq (T/n)(L^{1/2}f|L^{1/2}f)$, we finally get

$$\mathrm{P}\left(\sup_{j=0,\ldots,n} |f(X_{T_j})| \geq \Lambda\right) \leq \frac{\mathrm{e}}{\Lambda} \sqrt{(f|f) + T(L^{1/2}f|L^{1/2}f)} \tag{5.27}$$

follows. We take the limit $n \to \infty$ in the left hand side of (5.27). By the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \mathrm{P}\left(\sup_{j=0,\ldots,n} |f(X_{T_j})| \geq \Lambda\right) = \mathrm{P}\left(\lim_{n\to\infty} \sup_{j=0,\ldots,n} |f(X_{T_j})| \geq \Lambda\right).$$

Since $f(X_t)$ is continuous in t, $\lim_{n\to\infty} \sup_{j=0,\ldots,n} |f(X_{T_j})| = \sup_{0 \leq s \leq T} |f(X_s)|$ follows. This completes the proof of the proposition. $\square$

Proof of Lemma 5.5.

Let $f \in C^{\infty}(\mathbb{R}^3)$ such that

$$f(x) = \begin{cases} |x|, & |x| \geq T^{\lambda}, \\ \leq T^{\lambda}, & T^{\lambda} - 1 < |x| < T^{\lambda}, \\ 0, & |x| \leq T^{\lambda} - 1. \end{cases}$$

Since $\{x \mid f(x) \geq T^{\lambda}\} = \{x \mid |x| \geq T^{\lambda}\}$] we see that

$$\int \mathbb{1}_{A_T^c} d\mathrm{P} = \mathrm{P}\left(\sup_{|s| \leq T} |X_s| > T^{\lambda}\right) = \mathrm{P}\left(\sup_{|s| \leq T} |f(X_s)| > T^{\lambda}\right). \tag{5.28}$$

By Proposition 5.7 we have

$$\mathrm{P}\left(\sup_{|s|\le T}|f(X_s)|>T^\lambda\right)\le\frac{2\mathrm{e}}{T^\lambda}\sqrt{(f,f)+T(L^{1/2}f|L^{1/2}f)}. \tag{5.29}$$

We have

$$\begin{aligned}(L^{\frac12}f|L^{\frac12}f)= &\quad \mathrm{q}_0(f\psi_\mathrm{p},f\psi_\mathrm{p})+(f\psi_\mathrm{p}|Vf\psi_\mathrm{p})\\ \le &\quad C(\nabla f\psi_\mathrm{p}|\nabla f\psi_\mathrm{p})+(f\psi_\mathrm{p}|Vf\psi_\mathrm{p})\\ \le &\quad C'\|f\nabla\psi_\mathrm{p}\|^2+C''\|\nabla f\cdot\psi_\mathrm{p}\|^2+\|V^{\frac12}f\psi_\mathrm{p}\|^2.\end{aligned}$$

Using the fact that $\operatorname{supp} f\subset\{|x|\ge T^\lambda-1\}$, $\nabla f\in O(T^\lambda)$ and Lemma 3.11, we obtain

$$(f|f)+(L^{\frac12}f|L^{\frac12}f)\le C\mathrm{e}^{-\delta T^{\lambda(\delta+1)}}.$$

This completes the proof of the lemma. □

Appendix A. Proof of Proposition 3.12

In order to prove Proposition 3.12 we need several steps. Let $B(\mathbb{R}^3)$ denotes the Borel σ-field. For $0\le t_0\le\cdots\le t_n$ let the set function $\nu_{t_0,\dots,t_n}:\prod_{j=0}^n B(\mathbb{R}^3)\to\mathbb{R}$ be given by

$$\nu_{t_0,\dots,t_n}\left(\prod_{i=0}^n A_i\right)=(\mathbb{1}_{A_0}|\mathrm{e}^{-(t_1-t_0)L}\mathbb{1}_{A_1}\cdots\mathrm{e}^{-(t_n-t_{n-1})L}\mathbb{1}_{A_n}) \tag{A.30}$$

and for $0\le t$, $\nu_t:B(\mathbb{R}^3)\to\mathbb{R}$ by

$$\nu_t\,(A)=(\mathbb{1}|\mathrm{e}^{-tL}\mathbb{1}_A)=(\mathbb{1}|\mathbb{1}_A). \tag{A.31}$$

(Step 1) The family of set functions $\{\nu_\xi\}_{\xi\subset\mathbb{R},\#\xi<\infty}$ given by (A.30) and (A.31) satisfies the consistency condition:

$$\nu_{t_0,\dots,t_{n+m}}\left(\prod_{i=0}^n A_i\times\prod_{i=n+1}^{n+m}\mathbb{R}^3\right)=\nu_{t_0,\dots,t_n}\left(\prod_{i=0}^n A_i\right)$$

and by the Kolmogorov extension theorem [KS, Theorem 2.2] there exists a probability measure ν_∞ on $((\mathbb{R}^3)^{[0,\infty)},B((\mathbb{R}^3)^{[0,\infty)}))$ such that

$$\nu_t\,(A)=\mathbb{E}_{\nu_\infty}\left[\mathbb{1}_A(Y_t)\right], \tag{A.32}$$

$$\nu_{t_0,\dots,t_n}\left(\prod_{i=0}^n A_i\right)=\mathbb{E}_{\nu_\infty}\left[\prod_{j=0}^n\mathbb{1}_{A_j}(Y_{t_j})\right],\quad n\ge1, \tag{A.33}$$

where $B((\mathbb{R}^3)^{[0,\infty)})$ denotes the σ-field generated by cylinder sets, and $Y_t(\omega)=\omega(t)$, $\omega\in(\mathbb{R}^3)^{[0,\infty)}$, is the coordinate mapping process. Then the process $Y=(Y_t)_{t\ge0}$ on the probability space $((\mathbb{R}^3)^{[0,\infty)},B((\mathbb{R}^3)^{[0,\infty)}),\nu_\infty)$ satisfies that

$$(f_0|\mathrm{e}^{-(t_1-t_0)L}f_1\cdots\mathrm{e}^{-(t_n-t_{n-1})L}f_n)=\mathbb{E}_{\nu_\infty}\left[\prod_{j=0}^n f_j(Y_{t_j})\right], \tag{A.34}$$

$$(\mathbb{1}|f)=(\mathbb{1}|\mathrm{e}^{-tL}f)=\mathbb{E}_{\nu_\infty}\left[f(Y_t)\right]=\mathbb{E}_{\nu_\infty}\left[f(Y_0)\right] \tag{A.35}$$

for $f_j\in L^\infty(\mathbb{R}^3)$, $j=0,1,...,n$.

(Step 2) We now see that the process Y has a continuous version.

Lemma A.1. *The process Y on $((\mathbb{R}^3)^{[0,\infty)},B((\mathbb{R}^3)^{[0,\infty)}),\nu_\infty)$ has a continuous version.*

Proof. We note that by (A.34), (A.35) and Proposition 3.7, $E_{\nu_\infty}[|Y_t-Y_s|^{2n}]$ can be expressed in terms of the diffusion process $X^{\rm x}=(X^{\rm x}_t)_{t\ge0}$, being the solution of the stochastic differential equation:

$$X^{{\rm x},j}_t-X^{{\rm x},j}_s=\int_s^t b_j(X^{\rm x}_r)dr+\sum_{k=1}^3\int_s^t\sigma_{j,k}(X^{\rm x}_r)\cdot dB^k_r.\quad j=1,2,3. \tag{A.36}$$

Since

$$E_{\nu_\infty}[|Y_t-Y_s|^{2n}]=\sum_{j=1}^3\sum_{k=0}^{2n}\left[\begin{array}{c}2n\\k\end{array}\right](-1)^k\mathbb{E}_{\nu_\infty}\left[(Y^j_t)^{2n-k}(Y^j_s)^k\right],$$

the left hand side above can be express in terms of ${\rm e}^{-tL}$ as

$$\begin{aligned}&E_{\nu_\infty}[|Y_t-Y_s|^{2n}]\\&=\sum_{j=1}^3\sum_{k=0}^{2n}\left[\begin{array}{c}2n\\k\end{array}\right](-1)^k\left((x^j)^{2n-k}\psi_{\rm p}|{\rm e}^{-(t-s)K}(x^j)^k\psi_{\rm p}\right)_{L^2}{\rm e}^{(t-s)\inf\sigma(L)}.\end{aligned}$$

Furthermore by Feynman-Kac formula, i.e., Proposition 3.7, the right-hand side above can be expressed in terms of $X^{\rm x}=(X^{\rm x}_t)_{t\ge0}$ as

$$\begin{aligned}&\mathbb{E}_{\nu_\infty}[|Y_t-Y_s|^{2n}]\\&=\int d\mu_{\rm p}({\rm x})\mathbb{E}_{\mathcal{W}}\left[|X^{\rm x}_{t-s}-X^{\rm x}_0|^{2n}\psi_{\rm p}(X^{\rm x}_0)\psi_{\rm p}(X^{\rm x}_{t-s}){\rm e}^{-\int_0^{t-s}V(X^{\rm x}_r)dr}\right]{\rm e}^{(t-s)\inf\sigma(L)}.\end{aligned}$$

Since $V\ge0$,

$$\mathbb{E}_{\nu_\infty}[|Y_t-Y_s|^{2n}]\le\|\psi_{\rm p}\|^2_\infty{\rm e}^{(t-s)\inf\sigma(L)}\int d\mu_{\rm p}({\rm x})\mathbb{E}_{\mathcal{W}}\left[|X^{\rm x}_{t-s}-X^{\rm x}_0|^{2n}\right].$$

We next estimate $\mathbb{E}_{\mathcal{W}}\left[|X^{\rm x}_t-X^{\rm x}_s|^{2n}\right]$. Since $X^{{\rm x},j}_t$ is the solution to the stochastic differential equation (A.36), we have

$$\mathbb{E}_{\mathcal{W}}\left[|X^{{\rm x},j}_t-X^{{\rm x},j}_s|^{2n}\right]\le2^{2n-1}\mathbb{E}_{\mathcal{W}}\left[\frac{|t-s|^{2n}}{2^{2n}}\|b_j\|^{2n}_\infty+\sum_{k=1}^3\left|\int_s^t\sigma_{jk}(X^{\rm x}_r)dB^k_r\right|^{2n}\right].$$

By the Burkholder-Davies-Gundy inequality [KS, Theorem 3.28], we have

$$\mathbb{E}_{\mathcal{W}}\left[\left|\int_s^t\sigma_{jk}(X^{\rm x}_r)dB^k_r\right|^{2n}\right]\le(n(2n-1))^n|t-s|^n\|\sigma_{jk}\|^{2n}_\infty.$$

Then $\mathbb{E}_{\mathcal{W}}\left[|X^{\rm x}_t-X^{\rm x}_s|^{2n}\right]\le C|t-s|^n$ with some constant C independent of s and t, and

$$\mathbb{E}_{\nu_\infty}\left[|Y_t-Y_s|^{2n}\right]\le C|t-s|^n \tag{A.37}$$

follows. Thus $Y=(Y_t)_{t\ge0}$ has a continuous version by Kolmogorov-Čentov continuity theorem [KS, Theorem 2.8]. □

Let $\overline{Y}=(\overline{Y}_t)_{t\ge0}$ be the continuous version of Y on $((\mathbb{R}^3)^{[0,\infty)},B((\mathbb{R}^3)^{[0,\infty)}),\nu_\infty)$. The image measure of ν_∞ on $(\mathscr{X}_+,B(\mathscr{X}_+))$ with respect to $\overline{Y}$ is denoted by Q, i.e., ${\rm Q}=\nu_\infty\circ\overline{Y}^{-1}$, and $\widetilde{Y}_t(\omega)=\omega(t)$ for $\omega\in\mathscr{X}_+$ is the coordinate mapping process. Then we constructed a stochastic process $\widetilde{Y}=(\widetilde{Y}_t)_{t\ge0}$ on $(\mathscr{X}_+,B(\mathscr{X}_+),{\rm Q})$ such that $\bar{Y}\stackrel{\rm d}{=}\widetilde{Y}$. Then (A.34) and (A.35) can be expressed in terms of $\widetilde{Y}$ as

$$(f_0|{\rm e}^{-(t_1-t_0)L}f_1\cdots{\rm e}^{-(t_n-t_{n-1})L}f_n)=\mathbb{E}_{\rm Q}\left[\prod_{j=0}^n f_j(\widetilde{Y}_{t_j})\right],$$

$$(\mathbb{1}|f)=(\mathbb{1}|{\rm e}^{-tL}f)=\mathbb{E}_{\rm Q}\left[f(\widetilde{Y}_t)\right]=\mathbb{E}_{\rm Q}\left[f(\widetilde{Y}_0)\right].$$

(Step 3) Define the regular conditional probability measure on $\mathscr{X}_+$ by

$$\mathrm{Q}^{\mathrm{x}}(\cdot) = \mathrm{Q}(\cdot|\widetilde{Y}_0 = \mathrm{x}) \tag{A.38}$$

for each $\mathrm{x} \in \mathbb{R}^3$. It is well defined, since $\mathscr{X}_+$ is a Polish space (completely separable metrizable space). See e.g., [KS, Theorems 3.18. and 3.19]. Since the distribution of $\widetilde{Y}_0$ equals to $d\mu_{\mathrm{p}}(\mathrm{x})$, note that $\mathrm{Q}(A) = \int d\mu_{\mathrm{p}}(\mathrm{x})\mathbb{E}_{\mathrm{Q}^{\mathrm{x}}}[\mathbb{1}_A]$. Then the stochastic process $\tilde{Y} = (\tilde{Y}_t)_{t\geq 0}$ on $(\mathscr{X}_+, B(\mathscr{X}_+), \mathrm{Q}^{\mathrm{x}})$ satisfies

$$(f_0|\mathrm{e}^{-(t_1-t_0)L}f_1 \cdots \mathrm{e}^{-(t_n-t_{n-1})L}f_n) = \int d\mu_{\mathrm{p}}(\mathrm{x})\mathbb{E}_{\mathrm{Q}^{\mathrm{x}}}\left[\prod_{j=0}^n f_j(\widetilde{Y}_{t_j})\right], \tag{A.39}$$

$$(\mathbb{1}|\mathrm{e}^{-tL}f) = (\mathbb{1}|f) = \int dx\psi_{\mathrm{p}}^2(x)\mathbb{E}_{\mathrm{Q}^{\mathrm{x}}}\left[f(\widetilde{Y}_0)\right] = \int d\mu_{\mathrm{p}}(\mathrm{x})f(x). \tag{A.40}$$

Lemma A.2. *$\widetilde{Y}$ is a Markov process on $(\mathscr{X}_+, B(\mathscr{X}_+), \mathrm{Q}^{\mathrm{x}})$ with respect to the natural filtration $\mathscr{M}_s = \sigma(\widetilde{Y}_r, 0 \leq r \leq s)$.*

Proof. Let

$$p_t(\mathrm{x}, A) = \left(\mathrm{e}^{-tL}\mathbb{1}_A\right)(\mathrm{x}), \quad A \in B(\mathbb{R}^3), \quad t \geq 0. \tag{A.41}$$

Notice that $p_t(\mathrm{x}, A) = \mathbb{E}_{\mathcal{W}}\left[\mathbb{1}_A(X_t^{\mathrm{x}})\right]$. Then the finite dimensional distribution of $\tilde{Y}$ is

$$\mathbb{E}_{\mathrm{Q}^{\mathrm{x}}}\left[\prod_{j=1}^n \mathbb{1}_{A_j}(\widetilde{Y}_{t_j})\right] = \int \prod_{j=1}^n \mathbb{1}_{A_j}(\mathrm{x}_j) \prod_{j=1}^n p_{t_j-t_{j-1}}(\mathrm{x}_{j-1}, d\mathrm{x}_j) \tag{A.42}$$

with $t_0 = 0$ and $\mathrm{x}_0 = \mathrm{x}$ by (A.39). We show that $p_t(\mathrm{x}, A)$ is a probability transition kernel, i.e., (1) $p_t(\mathrm{x}, \cdot)$ is a probability measure on $B(\mathbb{R}^3)$, (2) $p_t(\cdot, A)$ is Borel measurable with respect to x, (3) the Chapman-Kolmogorov equality

$$\int p_s(\mathrm{y}, A)p_t(\mathrm{x}, d\mathrm{y}) = p_{s+t}(\mathrm{x}, A) \tag{A.43}$$

is satisfied. Note that e^{-tL} is positivity improving. Then $0 \leq \mathrm{e}^{-tL}f \leq \mathbb{1}$ for all function f such that $0 \leq f \leq \mathbb{1}$, and $\mathrm{e}^{-tL}\mathbb{1} = \mathbb{1}$ follows. Then $p_t(\mathrm{x}, \cdot)$ is the probability measure on $\mathbb{R}^3$ with $p_t(\mathrm{x}, \mathbb{R}^3) = 1$, and (1) follows. (2) is trivial. From the semi-group property $\mathrm{e}^{-sL}\mathrm{e}^{-tL}\mathbb{1}_A = \mathrm{e}^{-(s+t)L}\mathbb{1}_A$, the Chapman-Kolmogorov equality (A.43) follows. Hence $p_t(\mathrm{x}, A)$ is a probability transition kernel. We write $\mathbb{E}$ for $\mathbb{E}_{\mathrm{Q}^{\mathrm{x}}}$ for notational simplicity. From the identity $\mathbb{E}[\mathbb{1}_A(\widetilde{Y}_t)\mathbb{E}[f(\widetilde{Y}_r)|\sigma(\widetilde{Y}_t)]] = \mathbb{E}[\mathbb{1}_A(\widetilde{Y}_t)f(\widetilde{Y}_r)]$ for $r > t$, it follows that

$$\int \mathbb{1}_A(\mathrm{y})\mathbb{E}[f(\widetilde{Y}_r)|\widetilde{Y}_t = \mathrm{y}]P_t(d\mathrm{y}) = \int P_t(d\mathrm{y})\mathbb{1}_A(\mathrm{y}) \int f(\mathrm{y}')p_{r-t}(\mathrm{y}, d\mathrm{y}'),$$

where $P_t(d\mathrm{y})$ denotes the distribution of $\widetilde{Y}_t$ on $\mathbb{R}^3$. Thus

$$\mathbb{E}[f(\widetilde{Y}_r)|\widetilde{Y}_t = \mathrm{y}] = \int f(\mathrm{y}')p_{r-t}(\mathrm{y}, d\mathrm{y}')$$

follows a.e. y with respect to $P_t(d\mathrm{y})$. Then $\mathbb{E}[f(\widetilde{Y}_r)|\sigma(\widetilde{Y}_t)] = \int f(\mathrm{y})p_{r-t}(\widetilde{Y}_t, d\mathrm{y})$ and

$$\mathbb{E}[\mathbb{1}_A(\widetilde{Y}_r)|\sigma(\widetilde{Y}_t)] = p_{r-t}(\widetilde{Y}_t, A) \tag{A.44}$$

follow. By using (A.44), (A.42) and the Chapman-Kolmogorov equality (A.43), we can show that

$$\mathbb{E}\left[\mathbb{1}_A(\widetilde{Y}_{t+s}) \prod_{j=0}^n \mathbb{1}_{A_j}(\widetilde{Y}_{t_j})\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_A(\widetilde{Y}_t)|\sigma(\widetilde{Y}_s)\right] \prod_{j=0}^n \mathbb{1}_{A_j}(\widetilde{Y}_{t_j})\right]$$

for $t_0 \le \cdots \le t_n \le s$. This implies that $\mathbb{E}[\mathbb{1}_A(\widetilde{Y}_{t+s})|\mathscr{M}_s] = \mathbb{E}[\mathbb{1}_A(\widetilde{Y}_t)|\sigma(\widetilde{Y}_s)]$. Then $\widetilde{Y}$ is Markov with respect to the natural filtration under the measure Q^{x}. □

(Step 4) We extend $\widetilde{Y} = (\widetilde{Y}_t)_{t\ge 0}$ to a process on the whole real line $\mathbb{R}$. Set $\tilde{\mathscr{X}}_+ = \mathscr{X}_+ \times \mathscr{X}_+$, $\tilde{\mathscr{M}} = B(\mathscr{X}_+) \times B(\mathscr{X}_+)$ and $\tilde{\mathrm{Q}}^{\mathrm{x}} = \mathrm{Q}^{\mathrm{x}} \times \mathrm{Q}^{\mathrm{x}}$. Let $(\tilde{X}_t)_{t\in\mathbb{R}}$ be the stochastic process on the product space $(\tilde{\mathscr{X}}_+, \tilde{\mathscr{M}}, \tilde{\mathrm{Q}}^{\mathrm{x}})$, defined by for $\omega = (\omega_1, \omega_2) \in \tilde{\mathscr{X}}_+$,

$$\tilde{X}_t(\omega) = \begin{cases} \widetilde{Y}_t(\omega_1), & t \ge 0, \\ \widetilde{Y}_{-t}(\omega_2), & t < 0. \end{cases} \tag{A.45}$$

Note that $\tilde{X}_0 = \mathrm{x}$ almost surely with respect to $\tilde{\mathrm{Q}}^{\mathrm{x}}$ and $\tilde{X}_t$ is continuous in t almost surely. It is trivial to see that $\tilde{X}_t$, $t \ge 0$, and $\tilde{X}_s$, $s \le 0$, are independent, and $\tilde{X}_t \stackrel{\mathrm{d}}{=} \tilde{X}_{-t}$.

(Step 5) Proof of Theorem 3.12:
The image measure of $\tilde{\mathrm{Q}}^{\mathrm{x}}$ on $(\mathscr{X}, B(\mathscr{X}))$ with respect to $\tilde{X}$ is denoted by P^{x}, i.e.,

$$\mathrm{P}^{\mathrm{x}} = \tilde{\mathrm{Q}}^{\mathrm{x}} \circ \tilde{X}^{-1}. \tag{A.46}$$

Let $X_t(\omega) = \omega(t)$, $t \in \mathbb{R}$, $\omega \in \mathscr{X}$, be the coordinate mapping process. Then we can see that

$$X_t \stackrel{\mathrm{d}}{=} \tilde{Y}_t \quad (t \ge 0), \qquad X_t \stackrel{\mathrm{d}}{=} \tilde{Y}_{-t} \quad (t \le 0). \tag{A.47}$$

Since by (Step 3), $(\tilde{Y}_t)_{t\ge 0}$ and $(\tilde{Y}_{-t})_{t\le 0}$ are Markov processes with respect to the natural filtration $\sigma(\tilde{Y}_s, 0 \le s \le t)$ and $\sigma(\tilde{Y}_s, -t \le s \le 0)$, respectively, $(X_t)_{t\ge 0}$ and $(X_t)_{t\le 0}$ are also Markov processes with respect to $(\mathcal{F}_t^+)_{t\ge 0}$ and $(\mathcal{F}_t^-)_{t\le 0}$, respectively. Thus the Markov property (4) follows. We also see that $(X_s)_{s\le 0}$ and $(X_t)_{t\ge 0}$ are independent and $X_{-t} \stackrel{\mathrm{d}}{=} X_t$ by (A.47) and (Step 4). Thus reflection symmetry (3) follows.

Lemma A.3. *Let $-\infty < t_0 \le t_1 \le \cdots \le t_n$. Then*

$$\int d\mu_{\mathrm{p}}(\mathrm{x}) \mathbb{E}_{\mathrm{P}^{\mathrm{x}}} \left[f_0(X_{t_0}) \cdots f_n(X_{t_n})\right] = (f_0, \mathrm{e}^{-(t_1-t_0)L} f_1 \cdots \mathrm{e}^{-(t_n-t_{n-1})L} f_n). \tag{A.48}$$

Proof. Let $t_0 \le \cdots \le t_n \le 0 \le t_{n+1} \le \cdots t_{n+m}$. Then we have by the independence of $(X_s)_{s\le 0}$ and $(X_t)_{t\ge 0}$,

$$\begin{aligned} &\int d\mu_{\mathrm{p}}(\mathrm{x}) \mathbb{E}_{\mathrm{P}^{\mathrm{x}}} \left[f_0(X_{t_0}) \cdots f_{n+m}(X_{t_{n+m}})\right] \\ &= \int d\mu_{\mathrm{p}}(\mathrm{x}) \mathbb{E}_{\mathrm{P}^{\mathrm{x}}} \left[f_0(X_{t_0}) \cdots f_n(X_{t_n})\right] \mathbb{E}_{\mathrm{P}^{\mathrm{x}}} \left[f_{n+1}(X_{t_{n+1}}) \cdots f_{n+m}(X_{t_{n+m}})\right]. \end{aligned}$$

Since we have

$$\begin{aligned} &\mathbb{E}_{\mathrm{P}^{\mathrm{x}}} \left[f_{n+1}(X_{t_{n+1}}) \cdots f_{n+m}(X_{t_{n+m}})\right] \\ &\quad = \left(\mathrm{e}^{-t_{n+1}L} f_{n+1} \mathrm{e}^{-(t_{n+2}-t_{n+1})L} f_{n+2} \cdots \mathrm{e}^{-(t_{n+m}-t_{n+m-1})L} f_{n+m}\right)(x) \end{aligned} \tag{A.49}$$

and

$$\begin{aligned} &\mathbb{E}_{\mathrm{P}^{\mathrm{x}}} \left[f_0(X_{t_0}) \cdots f_n(X_{t_n})\right] \\ &= \mathbb{E}_{\mathrm{P}^{\mathrm{x}}} \left[f_0(\widetilde{Y}_{-t_0}) \cdots f_n(\widetilde{Y}_{-t_n})\right] \\ &= \left(\mathrm{e}^{+t_n L} f_n \mathrm{e}^{-(t_n-t_{n-1})L} f_{n-1} \cdots \mathrm{e}^{-(t_1-t_0)L} f_1\right)(x), \end{aligned} \tag{A.50}$$

by (A.49) and (A.50) we obtain that

$$\begin{aligned}&\int d\mu_{\rm p}({\rm x})\mathbb{E}_{\rm P^x}\left[f_0(X_{t_0})\cdots f_{n+m}(X_{t_{n+m}})\right]\\&=({\rm e}^{+t_nL}f_n\cdots{\rm e}^{-(t_1-t_0)L}f_1,{\rm e}^{-t_{n+1}L}f_{n+1}\cdots{\rm e}^{-(t_{n+m}-t_{n+m-1})L}f_{n+m})\\&=(f_1,{\rm e}^{-(t_1-t_0)L}f_2\cdots{\rm e}^{-(t_{n+m}-t_{n+m-1})L}f_{n+m}).\end{aligned}$$

Hence (A.48) follows. □

From Lemma A.3 it follows that for any $s\in\mathbb{R}$,

$$\int d\mu_{\rm p}({\rm x})\mathbb{E}_{\rm P^x}\left[\prod_{j=0}^n f_j(X_{t_j})\right]=\int d\mu_{\rm p}({\rm x})\mathbb{E}_{\rm P^x}\left[\prod_{j=0}^n f_j(X_{t_j+s})\right].$$

Hence shift invariance (5) is obtained. □

Acknowledgments:
FH acknowledges support of Grant-in-Aid for Science Research (B) 20340032 from JSPS and Grant-in-Aid for Challenging Exploratory Research 22654018 from JSPS, and is thankful to the hospitality of Université de Paris XI, where part of this work has been done.

References

[AHH] Arai, A., Hirokawa, M., Hiroshima, F.: On the absence of eigenvectors of Hamiltonians in a class of massless quantum field models without infrared cutoff, *J. Funct. Anal.* **168** (1999), 470–497.

[BFS] Bach, V., Fröhlich, J., Sigal, I.M.:, Quantum electrodynamics of confined non-relativistic particles, *Adv. Math.* **137** (1998), 299–395.

[BHLMS] Betz, V., Hiroshima F., , Lőrinczi, L., Minlos, R., Spohn, H.: Ground state properties of the Nelson Hamiltonian — a Gibbs measure-based approach, *Rev. Math. Phys.*, **14** (2002), 173–198.

[BF] Berg, C., Forst G.: *Potential Theory on Locally Compact Abelian Groups*, Springer, 1975.

[BD] Bruneau, L. , Dereziński, J.: , Pauli-Fierz Hamiltonians defined as quadratic forms, *Rep. Math. Phys.* **54** (2004), 169-199.

[D] Davies, E. B.: *Heat kernels and spectral theory*, Cambridge tracts in mathematics **92**, Cambridge University Press, 1989.

[DG1] Dereziński, J., Gérard, C.: Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians, *Rev. Math. Phys.* **11** (1999), 383-450.

[DG2] Dereziński, J., Gérard, C.: Scattering theory of infrared divergent Pauli-Fierz Hamiltonians, *Ann. Henri Poincaré*, **5** (2004), 523-578.

[EN] Engel, K. J., Nagel, R.: *One-parameter semi-groups for linear evolution equations*, GTM**194**, Springer, (2000).

[E] Evans, L.C.: *An introduction to stochastic differential equations*, On line version

[GGM] Georgescu, V., Gérard, C., Moeller J.: Spectral theory of massless Nelson models, *Commun. Math. Phys.* **249** (2004), 29-78.

[G] Gérard, C., On the existence of ground states for massless Pauli-Fierz Hamiltonians, *Ann. Henri Poincaré* **1** (2000), 443–459, *and* A remark on the paper: "On the existence of ground states for Hamiltonians", mp-arc 06-146 (2006).

[GHPS1] Gérard, C., Hiroshima, F., Panati, A., Suzuki, A.: Infrared Divergence of a Scalar Quantum Field Model on a Pseudo Riemannian Manifold, *Interdisciplinary Information Sciences* **15**, (2009) 399-421.

[GHPS2] Gérard, C., Hiroshima, F., Panati, A., Suzuki, A.: Infrared problem for the Nelson model with variable coefficients, preprint hal-00476310, 2010.

[H] Hirokawa, M.: Infrared catastrophe for Nelson's model-non-existence of ground state and soft-boson divergence-, *Publ. RIMS, Kyoto Univ.* **42**, (2006), 897–922.

[HHS] Hirokawa, M., Hiroshima F., Spohn, H.: Ground state for point particles interacting through a massless scalar field, *Adv. Math.* **191** (2005), 339–392.

[HS] Hiroshima, F., Sasaki I.: Enhanced binding of an N particle system interacting with a scalar field I, *Math. Z.* **259** (2008) 657–680.

[KS] Karatzas I., Shreve, I.: *Brownian motion and stochastic calculus*, GTM **113**, 1991.

[KM] Kato, T., Masuda, K.: Trotter's product formula for nonlinear semi-groups generated by sub-differentials of convex functions, *J. Math. Soc. Japan* **30** (1978), 169–178.

[KV] Kipnis, C., Varadhan, S.R.S.: Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions, *Commun. Math. Phys.***104** (1986), 1–19.

[LMS] Lőrinczi, J., Minlos, R., Spohn, H.: The infrared behavior in Nelson's model of a quantum particle coupled to a massless scalar field, *Ann. Henri Poincaré* **3** (2002), 1–28.

[N] Nelson, E.: Interaction of non-relativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.

[O] Øksendal, B.: *Stochastic Differential Equations*, Springer Verlag 2003.

[P] Panati, A.: Existence and non existence of a ground state for the massless Nelson model under binding condition, *Rep. Math. Phys.* **63** (2009), 305–330.

[PE] Porper, F. O. , Eidel'man, S. D.: Two sided estimates of fundamental solutions of second order parabolic equations and some applications, *Russian Math. Surveys*, **39** (1984), 119–178.

[RS] Reed, M. , Simon, B.: *Methods of Modern Mathematical Physics IV*, Academic press, (1978).

[RW] Rogers, L., Williams, D.: *Diffusions, Markov processes and martingales*, **Vol. 2**, Cambridge University Press (2000).

[Sa] Sasaki, I.: Ground state of the massless Nelson model in a non-Fock representation, *J. Math. Phys.* **46** (2005), 102107.

[Se] Semenov, Y. A. : Stability of L^p-spectrum of generalized Schrödinger operators and equivalence of Green functions, *IMRN* **12** (1997), 573–593.

[S] Simon, B. : *Functional Integration and Quantum Physics*, 2nd ed., AMS Chelsea Publishing, 2004.

[Sp] Spohn, H.: Ground state of quantum particle coupled to a scalar boson field, *Lett. Math. Phys.* **44** (1998), 9–16.

[SV] Stroock, D. W. , Varadhan, S. R. S. :*Multidimensional Diffusion Processes*, Grundlehren der mathematischen Wissenschaften **233**, Springer (2006).

[Zh] Zhang, Q. S.: Large time behavior of Schroedinger heat kernels and applications, *Commun. Math. Phys.* **210** (2000), 371-398.

Département de Mathématiques, Université de Paris XI, 91405 Orsay Cedex France
E-mail address: christian.gerard@math.u-psud.fr

Department of Mathematics, University of Kyushu, 6-10-1, Hakozaki, Fukuoka, 812-8581, Japan
E-mail address: hiroshima <hiroshima@math.kyushu-u.ac.jp>

PHYMAT, Université Toulon-Var 83957 La Garde Cedex France
E-mail address: annalisa.panati@univ-tln.fr

Department of Mathematics, Faculty of Engineering, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan
E-mail address: sakito@math.kyushu-u.ac.jp